Finding periodic orbits in state-dependent delay differential equations as roots of algebraic equations

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In this paper we prove that periodic boundary-value problems (BVPs) for delay differential equations are locally equivalent to finite-dimensional algebraic systems of equations. We rely only on regularity assumptions that follow those of the review by Hartung *et al.* (2006). Thus, the equivalence result can be applied to differential equations with state-dependent delays (SD-DDEs), transferring many results of bifurcation theory for periodic orbits to this class of systems. We demonstrate this by using the equivalence to give an elementary proof of the Hopf bifurcation theorem for differential equations with state-dependent delays. This is an alternative and extension to the original Hopf bifurcation theorem for SD-DDEs by Eichmann (2006).

1. Introduction

If a dynamical system is described by a differential equation where the derivative at the current time may depend on states in the past one speaks of delay differential or, more generally, functional differential equations (FDEs). A reasonably general formulation of an autonomous dynamical system of this type looks like this:

$$\dot{x}(t) = f(x_t, \mu) \tag{1}$$

where $\tau > 0$ is an upper bound for the delay. On the right-hand side f is a functional, mapping $C^0([-\tau, 0]; \mathbb{R}^n)$ (the space of continuous functions on the interval $[-\tau, 0]$ with values in $\mathbb{R}^n$) into $\mathbb{R}^n$. The dependent variable x is a function on $[-\tau, T_{\max})$ for some $T_{\max} > 0$, and x_t is the current *function segment*: $x_t(s) = x(t+s)$ for $s \in [-\tau, 0]$ such that $x_t \in C^0([-\tau, 0]; \mathbb{R}^n)$. The second argument $\mu \in \mathbb{R}^\nu$ is a system parameter. For a system of the form (1) one would have to prescribe a continuous function x on the interval $[-\tau, 0]$ as the initial value and then extend x toward time $T_{\max}$ (see textbooks on functional differential equations such as [4, 11, 22]).

A long-standing problem with certain types of FDEs is that they do not fit well into the general framework of smooth infinite-dimensional dynamical system theory. The problem occurs whenever the functional f invokes the evaluation operation in a non-trivial way, that is, for example, if one has a state-dependent delay. A prototypical caricature example would be the functional

$$f : U \times \mathbb{R} \mapsto \mathbb{R}, \qquad\qquad f(x,\mu) = \mu - x(-x(0)), \tag{2}$$

where $U = \{x \in C^0([-\tau,0];\mathbb{R}) : 0 < x(0) < \tau\}$ is an open set in $C^0([-\tau,0];\mathbb{R})$. The corresponding FDE is

$$\dot{x}(t) = \mu - x(t - x(t)). \tag{3}$$

Here, f evaluates its first argument x at a point that itself depends on x. We restrict ourselves to solutions x of (3) with $x(t) \in (0,\tau)$ for $t \geq 0$ to avoid problems with causality and to limit the maximal delay to τ (always keeping x_t in U).

The difficulty with (3) stems from the fact that f as a map is only as smooth as its argument x. Specifically, the derivative of f with respect to its first argument in this example exists only for $x \in C^1([-\tau,0];\mathbb{R})$ (the space of all continuously differentiable functions on $[-\tau,0]$):

$$\begin{aligned} &\partial^1 f : C^1([-\tau,0];\mathbb{R}) \times \mathbb{R} \times C^1([-\tau,0];\mathbb{R}) \mapsto \mathbb{R}, \\ &\partial^1 f(x,\mu,y) = x'(-x(0))\, y(0) - y(-x(0)). \end{aligned} \tag{4}$$

So, if we choose U as the phase space for initial-value problems (IVPs) in example (3) then the functional f is not differentiable for all elements of U. In fact, it is not even locally Lipschitz continuous in U. Indeed, Winston [25] gave an example of an initial condition in U for (3) (with $\mu = 0$ and $\tau > 1$), for which the IVP did not have a unique solution. This counterexample is not surprising since the right-hand side f does not fit into the framework that the textbooks [4, 11, 22] assume to be present. A result of Walther [24] rescues IVPs with state-dependent delays (such as (3)) by restricting the phase space in general to the closed submanifold C_c of $C^1([-\tau,0];\mathbb{R}^n)$:

$$C_c = \{x \in C^1([-\tau,0];\mathbb{R}^n) : x'(0) = f(x)\}.$$

Walther [24] could prove the existence of a semiflow inside this manifold that is continuously differentiable with respect to its initial conditions. However, this result is restricted to a single degree of differentiability. Results about higher degrees of smoothness are lacking for the semiflow [12].

A typical task one wants to perform for problems of type (1), or example (3), is bifurcation analysis of *equilibria* and *periodic orbits*. Equilibria are solutions x of (1) that are constant in time, and periodic orbits are solutions $x \in C^1(\mathbb{R};\mathbb{R}^n)$ of (1) that satisfy $x(t+T) = x(t)$ for some $T > 0$ and all $t \in \mathbb{R}$. Equilibria of the general FDE (1) can be determined by finding the solutions $(p,\mu) \in \mathbb{R}^n \times \mathbb{R}^\nu$ of the algebraic system of equations

$$0 = f(E_0 p, \mu) \tag{5}$$

where E_0 is the trivial embedding

$$E_0 : \mathbb{R}^n \mapsto C^0([-\tau, 0]; \mathbb{R}^n), \quad [E_0 p](s) = p \quad \text{for all } s \in [-\tau, 0].$$

We observe that, even though the FDE (1) is an infinite-dimensional system, its equilibria can be found as roots of the finite-dimensional system (5) of algebraic equations. Moreover, the regularity problems of the semiflow do not affect (5): in the example (3), the algebraic equation (5) reads $0 = \mu - p$, which is smooth to arbitrary degree, and can be solved even for negative μ (near equilibria with $\mu = p < 0$ the semiflow does not exist).

In this paper we establish a system similar to (5), but for periodic orbits: we find a finite-dimensional algebraic system of equations that does not suffer from the regularity problems affecting the semiflow, and an equivalence between solutions of this algebraic system and periodic orbits of (1). In comparison, for ordinary differential equations (ODEs) of the form $\dot{x}(t) = f(x(t), \mu)$ with a smooth $f : \mathbb{R}^n \times \mathbb{R}^\nu \mapsto \mathbb{R}^n$, the fact that the problem of finding periodic orbits can be reduced to algebraic root-finding is well known [9]. For example, in ODEs one can use the algebraic system $0 = X(T; p, \mu) - p$ where $t \mapsto X(t; p, \mu)$ is the trajectory defined by the IVP starting from $p \in \mathbb{R}^n$ and using parameter $\mu \in \mathbb{R}^\nu$.

A central notion in the construction of the equivalent algebraic system for periodic orbits of FDEs are *periodic boundary-value problems* (BVPs) for FDEs on the interval $[-\pi, \pi]$ with periodic boundary conditions (which we identify with the unit circle $\mathbb{T}$). Periodic orbits of (1) can then be found as solutions of periodic BVPs. If one wants to make the equivalence result useful in practical applications, one has to find a regularity (smoothness) condition on the right-hand side f that includes the class of state-dependent delay equations reviewed in [12], while still ensuring that it is possible to prove the existence of an equivalent algebraic system. We use exactly the same condition as used by Walther in [24] to prove the existence of a continuously differentiable semiflow, the so-called *extendable continuous differentiability* (originally introduced as "almost Frechét differentiability" in [21]), which implies a restricted form of local Lipschitz continuity. We generalize restricted continuous differentiability to higher degrees of restricted smoothness (which we call EC^k smoothness) in a similar fashion as Krisztin [19] did for the proof of the existence and smoothness of local unstable manifolds of equilibria. Our definition of EC^k smoothness is comparatively simple to state and check, and lends itself easily to inductive proofs.

After introducing the notation for periodic BVPs and EC^k smoothness we state the main result, an equivalence theorem between periodic BVPs and algebraic systems of equations in Section 2. The equivalence theorem reduces statements about existence and smooth dependence of periodic orbits of FDEs to root-finding problems of smooth algebraic equations. The result is weaker than the corresponding results for equilibria of FDEs and for periodic orbits of ODEs because the equivalence is only valid locally. For any given periodic function x_0 with Lipschitz continuous time derivative we construct an algebraic system that is equivalent to the periodic BVP in a sufficiently small open neighborhood of x_0. However, the result is still useful, as we then demonstrate in Section 3. We apply the equivalence theorem in the vicinity of equilibria for which the linearization of (1) has eigenvalues on the imaginary axis (for example, near $x_0 = \mu = \pi/2$ in example (3)) to prove the Hopf Bifurcation Theorem. The equivalence theorem reduces the proof of the Hopf Bifurcation

Theorem to an application of the Algebraic Branching Lemma [1]. This provides a complete proof for the Hopf Bifurcation Theorem for FDEs with state-dependent delays, including the regularity of the emerging periodic orbits. We discuss differences to the first version of the proof by Eichmann [5] and the approach of Hu and Wu [13] in Section 3. The equivalence is applicable in other scenarios where one would expect branching of periodic solutions. Examples are period doublings, the branching from periodic orbits with resonant Floquet multipliers on the unit circle in Arnol'd tongues, and branching scenarios in FDEs with symmetries. We give a tentative list of straightforward applications and generalizations of the equivalence theorem in the conclusion (Section 4).

We note that the theorem stated in Section 2 differs from statements about numerical approximations. As part of the theorem we also provide a map X that maps the root of the algebraic system back into a function space to give the *exact* solution of the periodic BVP, and a projection P that maps functions to finite-dimensional vectors (and, hence, periodic orbits to roots of the algebraic system). In numerical methods one typically has to increase the dimension of the algebraic system in order to get more and more accurate *approximations* of the true solution whereas the dimension of the algebraic system constructed in Section 2 is finite.

2. The Equivalence Theorem

This section states the assumptions and conclusions of the main result of the paper, the Equivalence Theorem stated in Theorem 2.5. Before doing so, we introduce some basic notation (function spaces on intervals with periodic boundary conditions and projections onto the leading Fourier modes).

Periodic BVPs

We first state precisely what we mean by periodic BVP and introduce the usual hierarchy of continuous, continuously differentiable and Lipschitz continuous functions on the compact interval $[-\pi,\pi]$ with periodic boundary conditions. For $j\geq 0$ we will use the notation $C^j(\mathbb{T};\mathbb{R}^n)$ for the spaces of all functions x on the interval $[-\pi,\pi]$ with continuous derivatives up to order j (including order 0 and j) satisfying the periodic boundary conditions $x^{(l)}(-\pi)=x^{(l)}(\pi)$ for $l=0\ldots j$. Elements of $C^0(\mathbb{T};\mathbb{R}^n)$ are continuous and satisfy $x(-\pi)=x(\pi)$. For derivatives of order $j>0$, $x^{(j)}(-\pi)$ is the right-sided jth derivative of x in $-\pi$, and $x^{(j)}(\pi)$ is the left-sided jth derivative of x in π. The norm in $C^j(\mathbb{T};\mathbb{R}^n)$ is

$$\|x\|_j=\max_{t\in[-\pi,\pi]}\left\{|x(t)|,|x'(t)|,\ldots,|x^{(j)}(t)|\right\}.$$

We can extend any function x in $C^j(\mathbb{T};\mathbb{R}^n)$ to arguments in $\mathbb{R}$ by defining $x(t)=x(t-2k\pi)$ where k is an integer chosen such that $-\pi\leq t-2k\pi<\pi$ (we will write $t_{\mathrm{mod}[-\pi,\pi)}$ later). Thus, every element of $C^j(\mathbb{T};\mathbb{R}^n)$ is also an element of $BC^j(\mathbb{R};\mathbb{R}^n)$, the space of functions with bounded continuous derivatives up to order j on the real line. We use the notation $t\in\mathbb{T}$ for arguments t of x, and also call $\mathbb{T}$ the unit circle. This make sense because the

parametrization of the unit circle by angle provides a cover, identifying $\mathbb{T}$ with $\mathbb{R}$ where we use $[-\pi, \pi)$ as the fundamental interval.

Additional useful function spaces are the space of Lipschitz continuous functions and, correspondingly, spaces with Lipschitz continuous derivatives, denoted by $C^{j,1}(\mathbb{T};\mathbb{R}^n)$, which are equipped with the norm

$$\|x\|_{j,1} = \max\left\{\|x\|_j, \sup_{\substack{t,s\in\mathbb{R}\\ t\neq s}} \frac{|x^{(j)}(s) - x^{(j)}(t)|}{|s-t|}\right\} \tag{6}$$

($x^{(0)}(t)$ refers to $x(t)$). Note that we used the notation $t, s \in \mathbb{R}$ in the index of the supremum, as we can apply arbitrary arguments in $\mathbb{R}$ to a function $x \in C^0(\mathbb{T};\mathbb{R}^n)$ by considering it as an element of $BC^0(\mathbb{R};\mathbb{R}^n)$, as explained above. We use the same notation ($C^j(J;\mathbb{R}^n)$ and $C^{j,1}(J;\mathbb{R}^n)$) also for functions on an arbitrary compact interval $J \subset \mathbb{R}$ without periodic boundary conditions (and one-sided derivatives at the boundaries). As any function $x \in C^j(\mathbb{T};\mathbb{R}^n)$ is also an element of $BC^j(\mathbb{R};\mathbb{R}^n)$, it is also an element of $C^j(J;\mathbb{R}^n)$ for any compact interval J (and the norm of the embedding operator equals unity). On the function spaces $C^j(\mathbb{T};\mathbb{R}^n)$ we define the time shift operator

$$\Delta_t : C^j(\mathbb{T};\mathbb{R}^n) \mapsto C^j(\mathbb{T};\mathbb{R}^n), \qquad [\Delta_t x](s) = x(t+s).$$

The operator Δ_t is linear and has norm 1 in all spaces $C^j(\mathbb{T};\mathbb{R}^n)$. Similarly, Δ_t maps also $C^{j,1}(\mathbb{T};\mathbb{R}^n) \mapsto C^{j,1}(\mathbb{T};\mathbb{R}^n)$, and has unit norm there as well.

Let f be a continuous functional on the space of continuous periodic functions, that is,

$$f : C^0(\mathbb{T};\mathbb{R}^n) \mapsto \mathbb{R}^n.$$

The right-hand side f, together with the shift Δ_t, creates an operator in $C^0(\mathbb{T};\mathbb{R}^n)$, defined as

$$F : C^0(\mathbb{T};\mathbb{R}^n) \mapsto C^0(\mathbb{T};\mathbb{R}^n) \qquad\qquad [F(x)](t) = f(\Delta_t x). \tag{7}$$

The operator F is invariant with respect to time shift by construction: $F(\Delta_t x) = \Delta_t F(x)$. We consider autonomous periodic boundary-value problems for differential equations where f is the right-hand side:

$$\dot{x}(t) = f(\Delta_t x) = F(x)(t). \tag{8}$$

A function $x \in \mathbb{C}^1(\mathbb{T};\mathbb{R}^n)$ is a solution of (8) if x satisfies equation (8) for all $t \in \mathbb{T}$ (for each $t \in \mathbb{T}$ equation (8) is an equation in $\mathbb{R}^n$). In contrast to the introduction we do not expressly include a parameter μ as an argument of f. This does not reduce generality as we will explain in Section 3. The main result, the Equivalence Theorem 2.5, will be concerned with equivalence of the periodic BVP (8) to an algebraic system of equations. The notion of the shift Δ_t on the unit circle and the operator F, combining f with the shift, is specific to periodic BVPs such that the BVP (8) looks different from the IVP (1) in the introduction. Several results stating how regularity of f transfers to regularity of F are collected in Appendix A.

Definition of EC^k smoothness and local (restricted) EC Lipschitz continuity

Continuity of the functional f is not strong enough as a condition to prove the Equivalence Theorem. Rather, we need a notion of smoothness for f. However, as explained in the introduction, we cannot assume that f is continuously differentiable with degree $k \geq 1$, if we want to include examples such as $f(x) = -x(-x(0))$ (see FDE (3) for $\mu = 0$) into the class under consideration.

The review by Hartung *et al.* [12] observed the following typical property of functionals f appearing in equations of type (8): the derivative $\partial^1 f(x)$ of f in x as a linear map from $C^1(\mathbb{T};\mathbb{R}^n)$ into $\mathbb{R}^n$ can be extended to a bounded linear map from $C^0(\mathbb{T};\mathbb{R}^n)$ into $\mathbb{R}^n$, and the mapping

$$\partial^1 f : C^1(\mathbb{T};\mathbb{R}^n) \times C^0(\mathbb{T};\mathbb{R}^n) \mapsto \mathbb{R}^n \quad \text{defined by} \quad (x,y) \mapsto \partial^1 f(x,y)$$

is continuous as a function of both arguments. In other words, the derivative of f may depend on x' but not on y'. For the example $f(x) = -x(-x(0))$ this is true (see (4)). Most of the fundamental results establishing basic dynamical systems properties for FDEs with state-dependent delay in [12] rest on this extendability of $\partial^1 f$.

We also rely strongly on this notion of *extendable* continuous differentiability. The precise definition is given below in Definition 2.1. In this definition we permit the argument range J to be any compact interval or $\mathbb{T}$. We use the notation of a subspace of higher-order continuous differentiability not only for $C^j(J;\mathbb{R}^n)$ but also for products of such spaces in a natural way. Say, if

$$D = C^{k_1}(J;\mathbb{R}^{m_1}) \times \ldots \times C^{k_\ell}(J;\mathbb{R}^{m_\ell}), \tag{9}$$

where $\ell \geq 1$, and $k_j \geq 0$ and $m_j \geq 1$ are integers, and denoting the natural maximum norm on the product D by

$$\|x\|_D = \|(x_1,\ldots,x_\ell)\|_D = \max_{j\in\{1,\ldots,\ell\}} \|x_j\|_{k_j},$$

then for integers $r \geq 0$ the space D^r is defined in the natural way as

$$\begin{aligned} D^r &= C^{k_1+r}(J;\mathbb{R}^{m_1}) \times \ldots \times C^{k_\ell+r}(J;\mathbb{R}^{m_\ell}), \quad \text{with} \\ \|x\|_{D,r} &= \max_{\substack{0\leq j\leq r\\ 1\leq i\leq \ell}} \|x_i^{(j)}\|_{k_i}. \end{aligned}$$

For the simplest example, $D = C^0(J;\mathbb{R}^n)$, D^k is $C^k(J;\mathbb{R}^n)$. If $J = \mathbb{T}$ then the time shift Δ_t extends naturally to products of spaces:

$$\Delta_t x = (\Delta_t x_1,\ldots,\Delta_t x_\ell) \quad \text{for } x = (x_1,\ldots,x_\ell) \in D.$$

Definition 2.1 (Extendable continuous differentiability EC^k)

> Let D be a product space of the type (9), and let $f : D \mapsto \mathbb{R}^n$ be continuous. We say that f has an extendable continuous derivative if there exists a map $\partial^1 f$
>
> $$\partial^1 f : D^1 \times D \mapsto \mathbb{R}^n$$

that is continuous in both arguments $(u,v) \in D^1 \times D$ and linear in its second argument $v \in D$, such that for all $u \in D^1$

$$\lim_{\substack{v \in D^1 \\ \|v\|_{D,1} \to 0}} \frac{|f(u+v) - f(u) - \partial_1 f(u,v)|}{\|v\|_{D,1}} = 0. \tag{10}$$

We say that f is k times continuously differentiable in this extendable sense if the map $\partial^k f$, recursively defined as $\partial^k f = \partial^1[\partial^{k-1} f]$, exists and satisfies the limit condition (10) for $\partial^{k-1} f$. We abbreviate this notion by saying that f is EC^k smooth in D.

The limit in (10) is a limit in $\mathbb{R}$. For $k=1$ the definition is identical to property (S) in the review [12], one of the central assumptions for fundamental results on the semiflow. Extendable continuous differentiability requires the derivative to exist only in points in D^1 and with respect to deviations in D^1, but it demands that the derivative must extend in its second argument to D ($\partial^1 f$ is linear in its second argument). This is the motivation for calling this property *extendable* continuous differentiability.

The definition of EC^k smoothness for $k > 1$ uses the notation that a functional (say, $\partial^1 f$) of two arguments (say, $u \in D^1$ and $v \in D$) for which one would write $\partial^1 f(u,v)$, is also a functional of a single argument $w = (u,v) \in D^1 \times D$, such that one can also write $\partial^1 f(w)$. When using this notation we observe that the space $D^1 \times D$ is again a product of type (9) such that $\partial^1 f$ is again a functional of the same structure as f. For example, let us consider the functional $f : x \mapsto -x(-x(0))$ from example (2) (setting $\mu = 0$). The functional is well defined and continuous also on $D = C^0(\mathbb{T};\mathbb{R})$. Moreover, f is EC^k smooth in D to arbitrary degree k. Its first two derivatives are:

$$\begin{aligned}
&\partial^1 f : C^1(\mathbb{T};\mathbb{R}) \times C^0(\mathbb{T};\mathbb{R}), \\
&\partial^1 f(u,v) = u'(-u(0))\,v(0) - v(-u(0)), \text{ and}
\end{aligned}\tag{11}$$

$$\begin{aligned}
&\partial^2 f : \left[C^2(\mathbb{T};\mathbb{R}) \times C^1(\mathbb{T};\mathbb{R})\right] \times \left[C^1(\mathbb{T};\mathbb{R}) \times C^0(\mathbb{T};\mathbb{R})\right], \\
&\partial_2 f(u,v,w,x) = -\,u''(-u(0))\,w(0)\,v(0) + u'(-u(0))\,x(0) \\
&\qquad\qquad + w'(-u(0))\,v(0) - v'(-u(0))\,w(0) - x(-u(0)).
\end{aligned}\tag{12}$$

As one can see, the first derivative $\partial^1 f$ has the same structure as f itself if we replace $D = C^0(\mathbb{T};\mathbb{R}^n)$ by $D^1 \times D$. So, it is natural to apply the definition again to $\partial^1 f$ on the space $D^1 \times D$.

Assuming that f is EC^1 smooth on $C^0(J;\mathbb{R}^n)$ implies classical continuous differentiability of f as a map from $C^1(J;\mathbb{R}^n)$ into $\mathbb{R}^n$ and is, thus, strictly stronger than assuming that f is continuously differentiable on $C^1(\mathbb{T};\mathbb{R}^n)$.

Since every element of $C^j(\mathbb{T};\mathbb{R}^n)$ is also an element of $C^j(J;\mathbb{R}^n)$ for any compact interval J (and the embedding operator has unit norm), any EC^k smooth functional $f : C^0(J;\mathbb{R}^m) \mapsto \mathbb{R}^n$ is also a EC^k smooth functional from $C^0(\mathbb{T};\mathbb{R}^m)$ into $\mathbb{R}^n$.

It is worth comparing Definition 2.1 with the definition for higher degree of regularity used by Krisztin in [19]. With Definition 2.1 the kth derivative has 2^k arguments. In contrast to this, the kth derivative as defined in [19] has only $k+1$ arguments (the

first argument is the base point, and the derivative is a k-linear form in the other k arguments). The origin of this difference can be understood by looking at the example $f(x) = -x(-x(0))$ and its derivatives in (11)–(12). Krisztin's definition applied to the second derivative does not include the derivative of $\partial^1 f$ with respect to the linear second argument v (as is often convention, because it is the identity). One would obtain the second derivative according to Krisztin's definition by setting $x = 0$ in (12). Indeed, the terms containing the argument x in (12) are simply $\partial^1 f(u, x)$, as one expects when differentiating $\partial^1 f(u, v)$ with respect to v, calling the deviation x. While in practical examples it is often more economical to use the compact notation with k-forms, inductive proofs of higher-order differentiability using the full derivative only require the notion of at most bi-linear forms, making them less complex.

If f is EC^1 smooth then it automatically satisfies a restricted form of local Lipschitz continuity [12], which we call local EC Lipschitz continuity:

Definition 2.2 (Local EC Lipschitz continuity)

> We say that $f : C^0(\mathbb{T};\mathbb{R}^n) \mapsto \mathbb{R}^n$ is locally EC Lipschitz continuous if for every $x_0 \in C^1(\mathbb{T};\mathbb{R}^n)$ there exists a neighborhood $U(x_0) \subset C^1(\mathbb{T};\mathbb{R}^n)$ and a constant K such that
> $$|f(y) - f(z)| \leq K\|y - z\|_0 \tag{13}$$
> holds for all y and z in $U(x_0)$.

That EC^1 smoothness implies local EC Lipschitz continuity has been shown, for example, in [24] (but see also Lemma A.3 in Appendix A). Note that the estimate (13) uses the $\|\cdot\|_0$-norm for the upper bound. This is a sharper estimate than one would obtain using the expected $\|\cdot\|_1$-norm. The constant K may depend on the derivatives of the elements in $U(x_0)$ though. For example, for $f(x) = -x(-x(0))$ as in (2) with $\mu = 0$, one would have the estimate

$$\left|f(x+y) - f(x)\right| \leq \left[1 + \|x'\|_0\right] \|y\|_0 \quad \text{such that} \quad K \leq 1 + \max_{x \in U(x_0)} \|x\|_1.$$

This means that in this example, the neighborhood $U(x_0)$ can be chosen arbitrarily large as long as it is bounded in $C^1(\mathbb{T};\mathbb{R}^n)$.

The following lemma states that we can extend the neighborhood $U(x)$ in Definition 2.2 into the space of Lipschitz continuous functions ($C^{0,1}$ instead of C^1) and include time shifts (which possibly increases the bound K).

Lemma 2.3 (EC Lipschitz continuity uniform in time)

> Let f be locally EC Lipschitz continuous, and let x_0 be in $C^{0,1}(\mathbb{T};\mathbb{R}^n)$. Then there exists a bounded neighborhood $U(x_0) \subset C^{0,1}(\mathbb{T};\mathbb{R}^n)$ and a constant K such that
> $$|f(\Delta_t y) - f(\Delta_t z)| \leq K\|\Delta_t y - \Delta_t z\|_0 = K\|y - z\|_0$$
> holds for all y and z in $U(x_0)$, and for all $t \in \mathbb{T}$. Thus, $\|F(y) - F(z)\|_0 \leq K\|y - z\|_0$ for all y and z in $U(x_0)$.

Recall that $F(x)(t) = f(\Delta_t x)$. See Lemma A.3 and Lemma A.5 in Appendix A for the proof of Lemma 2.3. A consequence of Lemma 2.3 is that the time derivative of a solution x_0 of the periodic BVP is also Lipschitz continuous (in time): if $\dot{x}_0(t) = f(\Delta_t x_0)$ then there exists a constant K such that

$$\|x_0'(t) - x_0'(s)\|_0 \le K|t-s| \tag{14}$$

Thus, $x_0 \in C^{1,1}(\mathbb{T};\mathbb{R}^n)$. This follows from Lemma 2.3 by inserting $\Delta_t x_0$ and $\Delta_s x_0$ for y and z and using that $x_0'(t) = f(\Delta_t x_0)$ (it is enough to show (14) for $|t-s|$ small).

Projections onto subspaces spanned by Fourier modes

The variables of the algebraic system in the Equivalence Theorem will be the coefficients of the first N Fourier modes (where N will be determined as sufficiently large later) of elements of $C^{0,1}(\mathbb{T};\mathbb{R}^n)$ (the space of Lipschitz continuous functions on $\mathbb{T}$). Consider the functions on $\mathbb{T}$

$$b_0 = t \mapsto \frac{1}{2}, \quad b_k = t \mapsto \cos(kt), \quad b_{-k} = t \mapsto \sin(kt)$$

for $k = 1, \ldots, \infty$ (which is the classical Fourier basis of $\mathbb{L}^2(\mathbb{T};\mathbb{R})$). For any $m \ge 1$ we define the projectors and maps

$$\begin{aligned}
&P_N : C^j(\mathbb{T};\mathbb{R}^m) \mapsto C^j(\mathbb{T};\mathbb{R}^m), \quad [P_N x](t)_i = \sum_{k=-N}^{N} \left[\frac{1}{\pi}\int_{-\pi}^{\pi} b_k(s) x_i(s)\,\mathrm{d}s\right] b_k(t), \\
&Q_N = I - P_N, \\
&E_N : \mathbb{R}^{m\times(2N+1)} \mapsto C^j(\mathbb{T};\mathbb{R}^m), \quad [E_N p](t)_i = \sum_{k=-N}^{N} p_{i,k} b_k(t), \\
&R_N : C^j(\mathbb{T};\mathbb{R}^m) \mapsto \mathbb{R}^{m\times(2N+1)}, \quad [R_N x]_{i,k} = \frac{1}{\pi}\int_{-\pi}^{\pi} b_k(s) x_i(s)\,\mathrm{d}s, \\
&L \;: C^j(\mathbb{T};\mathbb{R}^m) \mapsto C^j(\mathbb{T};\mathbb{R}^m), \quad [Lx](t) = \int_0^t x(s) - R_0 x\;\mathrm{d}s = \int_0^t Q_0[x](s)\,\mathrm{d}s.
\end{aligned} \tag{15}$$

The projector P_N projects a periodic function onto the subspace spanned by the first $2N+1$ Fourier modes, and Q_N is its complement. The map E_N maps a vector p of $2N+1$ Fourier coefficients (which are each vectors of length n themselves) to the periodic function that has these Fourier coefficients. The map R_N extracts the first $2N+1$ Fourier coefficients from a function. The simple relation $P_N = E_N R_N$ holds. The vector $R_0 x$ is the average of a function x, and Q_0 subtracts the average from a periodic function. The operator L takes the anti-derivative of a periodic function after subtracting its average (to ensure that L maps back into the space of periodic functions). In all of the definitions the degree j of smoothness of the vector space C^j can be any non-negative integer. The operator L not only maps C^j back into itself, but it maps $C^j(\mathbb{T};\mathbb{R}^m)$ into $C^{j+1}(\mathbb{T};\mathbb{R}^m)$.

We do not attach an index m to the various maps to indicate how many dimensions the argument and, hence, the value has because there is no room for confusion: for example, if $x \in C^0(\mathbb{T};\mathbb{R}^2)$ then $P_N x \in C^0(\mathbb{T};\mathbb{R}^2)$ such that we use the same notation $P_N x$ for $x : \mathbb{T} \mapsto \mathbb{R}^m$ with arbitrary m. Similarly, we apply all maps also on product spaces D of the type $C^{k_1}(\mathbb{T};\mathbb{R}^{m_1}) \times \ldots \times C^{k_\ell}(\mathbb{T};\mathbb{R}^{m_\ell})$ introduced in Equation (9) by applying the maps element-wise. For example,

$$\begin{aligned} P_N x &= (P_N x_1, \ldots, P_N x_\ell) && \text{for } x = (x_1, \ldots, x_\ell) \in D, \\ E_N p &= (E_N p_1, \ldots, E_N p_\ell) && \text{for } p = (p_1, \ldots, p_\ell) \in \mathbb{R}^{m_1 \times (2N+1)} \times \ldots \times \mathbb{R}^{m_\ell \times (2N+1)}. \end{aligned}$$

Equivalent integral equation

We note the fact that a function $x \in C^1(\mathbb{T};\mathbb{R}^n)$ solves the periodic BVP $\dot{x}(t) = f(\Delta_t x) = F(x)(t)$ if and only if it satisfies the equivalent integral equation

$$x(t) = x(0) + \int_0^t F(x)(s)\,\mathrm{d}s \quad \text{for all } t \in \mathbb{T}. \tag{16}$$

For each $t \in \mathbb{T}$, Equation (16) is an equation in $\mathbb{R}^n$. In particular, the term $x(0)$ is in $\mathbb{R}^n$. Thus, the integral equation (16) is very similar to the corresponding integral equation used in the proof of the Picard-Lindelöf Theorem for ODEs [3]. This is in contrast to the abstract integral equations used by Diekmann *et al.* [4] to construct unique solutions to IVPs, in which equality at every point in time is an equality in function spaces. It is the similarity of (16) to its ODE equivalent that makes the reduction of periodic BVPs to finite dimensional algebraic equations possible. One minor problem is that the Picard iteration for (16) cannot be expected to converge. In fact, the integral term $\int_0^t F(x)(s)\,\mathrm{d}s$ does not even map back into the space $C^0(\mathbb{T};\mathbb{R}^n)$ of periodic functions, even if x is in $C^0(\mathbb{T};\mathbb{R}^n)$. However, a simple algebraic manipulation using the newly introduced maps L, P_N, Q_N, E_N and R_N removes this problem (remember that $F(x)(t) = f(\Delta_t x)$):

Lemma 2.4 (Splitting of BVP)

Let $N \geq 0$ be an arbitrary integer. A function $x \in C^0(\mathbb{T};\mathbb{R}^n)$ and a vector $p \in \mathbb{R}^{n\times(2N+1)}$ satisfy

$$\dot{x}(t) = f(\Delta_t x) \quad \text{and} \quad p = R_N x, \tag{17}$$

if and only if they satisfy the system

$$x = E_N p + Q_N L F(x), \tag{18}$$

$$0 = R_N \left[P_0 F(x) + Q_0 \left(E_N p - P_N L F(x)\right)\right]. \tag{19}$$

Note that the map R_N extracts the lowest $2N+1$ Fourier coefficients from a periodic function. Equation (18) can be viewed as a fixed-point equation in $C^{0,1}(\mathbb{T};\mathbb{R}^n)$, parametrized by p. We will apply the Picard iteration to this fixed-point equation instead of (16). Equation (19)

is an equation in $\mathbb{R}^{n\times(2N+1)}$. If the Picard iteration converges then the fixed-point equation (18) can be used to construct (for sufficiently large N) a map $X : U \subset \mathbb{R}^{n\times(2N+1)} \mapsto C^{0,1}(\mathbb{T};\mathbb{R}^n)$, which maps the parameter p to its corresponding fixed point x. Inserting this fixed point $x = X(p)$ into (19) turns (19) into a system of $n\times(2N+1)$ algebraic equations for the $n\times(2N+1)$-dimensional variable p, making the periodic BVP for x equivalent to an algebraic system for its first $2N+1$ Fourier coefficients, p. The proof of Lemma 2.4 is simple algebra, see Section 5.2.

Statement of the Equivalence Theorem

Using the Splitting Lemma 2.4 we can now state the central result of the paper. The intention to treat (18) as a fixed-point equation motivates the introduction of the map

$$\begin{aligned}&M_N : C^{0,1}(\mathbb{T};\mathbb{R}^n)\times\mathbb{R}^{n\times(2N+1)} \mapsto C^{0,1}(\mathbb{T};\mathbb{R}^n) \quad \text{given by}\\ &M_N(x,p) = E_N p + Q_N LF(x).\end{aligned}$$

This means that we will look for fixed points of the map $M_N(\cdot,p)$ for given p and sufficiently large N. We will do this in small closed balls in $C^{0,1}(\mathbb{T};\mathbb{R}^n)$ (the space of Lipschitz continuous functions) such that it is useful to introduce the notation

$$B_\delta^{0,1}(x_0) = \left\{x\in C^{0,1}(\mathbb{T};\mathbb{R}^n) : \left\|x-x_0\right\|_{0,1} \le \delta\right\},$$

for $\delta > 0$ and $x_0 \in C^{0,1}(\mathbb{T};\mathbb{R}^n)$. That is, $B_\delta^{0,1}(x_0)$ is the closed ball of radius δ around $x_0 \in C^{0,1}(\mathbb{T};\mathbb{R}^n)$ in the $\|\cdot\|_{0,1}$-norm (the Lipschitz norm on $\mathbb{T}$).

Theorem 2.5 (Equivalence between periodic BVPs and algebraic systems of equations)

> Let f be $EC^{j_{\max}}$ smooth, and let x_0 have a Lipschitz continuous derivative, that is, $x_0 \in C^{1,1}(\mathbb{T};\mathbb{R}^n)$. Then there exist a $\delta > 0$ and a positive integer N such that the map $M_N(\cdot,p)$ has a unique fixed point in $B_{6\delta}^{0,1}(x_0)$ for all p in the neighborhood U of $R_N x_0$ given by
>
> $$U = \left\{p\in\mathbb{R}^{n\times(2N+1)} : \left\|E_N\left[p - R_N x_0\right]\right\|_{0,1} \le 2\delta\right\}.$$
>
> The maps
>
> $$\begin{aligned}&X : U \mapsto C^0(\mathbb{T};\mathbb{R}^n), && X(p) = \text{ fixed point of } M_N(\cdot,p) \text{ in } B_{6\delta}^{0,1}(x_0),\\ &g : U \mapsto \mathbb{R}^{n\times(2N+1)}, && g(p) = R_N\left[P_0F(X(p)) + Q_0\left(E_Np - P_NLF(X(p))\right)\right],\end{aligned}$$
>
> are $j_{\max}$ times continuously differentiable with respect to their argument p, and $X(p)$ is an element of $C^{j_{\max}+1}(\mathbb{T};\mathbb{R}^n)$. Moreover, for all $x\in B_\delta^{0,1}(x_0)$ the following equivalence holds:
>
> $$\dot{x}(t) = f(\Delta_t x)$$
>
> if and only if $p = R_N x$ is in U and satisfies
>
> $$g(p) = 0 \quad \text{and} \quad x = X(p).$$

Theorem 2.5 is the central result of the paper. It implies that, for any $x_0 \in C^{1,1}(\mathbb{T};\mathbb{R}^n)$ all solutions of the periodic BVP in a sufficiently small neighborhood of x_0 lie in the graph $X(U)$ of a finite-dimensional manifold. Moreover, these solutions can be determined by finding the roots of g in $U \subset \mathbb{R}^{n\times(2N+1)}$. We note that Theorem 2.5 is different from statements about numerical approximations. Even though the integer N is finite, solving the algebraic system $g(p) = 0$ and then mapping the solutions with the map X into the function space $C^0(\mathbb{T};\mathbb{R}^n)$ gives an exact solution $x = X(p)$ of the periodic BVP $\dot{x}(t) = f(\Delta_t x)$.

The size of the radius δ of the ball in which the equivalence holds depends on how large one can choose δ such that a local EC Lipschitz constant K for F exists for $B^{0,1}_{6\delta}(x_0)$ (such neighborhoods exist according to Lemma 2.3). In many applications (in particular, in the example (3)) this can be any closed ball in which the right-hand side f is well defined (at the expense of increasing K for larger balls). Once the local EC Lipschitz constant K is determined, one can find a uniform upper bound R for the norm $\|F(x)\|_{0,1}$ for all $x \in B^{0,1}_{6\delta}(x_0)$ (see Lemma A.5). The integer N, which determines the dimension of the algebraic system, is then chosen depending on R, K and $\|x_0'\|_{0,1}$.

Section 5 contains the complete proof of Theorem 2.5. The first step of the proof of Equivalence Theorem 2.5 is the existence of the fixed point of M_N in $B^{0,1}_{6\delta}(x_0)$ for $p \in U$. This is achieved by applying Banach's contraction mapping principle to the map $M_N(\cdot, p)$ in the closed ball $B^{0,1}_{6\delta}(x_0)$. The only peculiarity in this step is that we apply the principle to $B^{0,1}_{6\delta}(x_0)$, which is a closed bounded set of Lipschitz continuous functions, using the (weaker) maximum norm ($\|\cdot\|_0$). This is possible because closed balls in $C^{0,1}(\mathbb{T};\mathbb{R}^n)$ are complete also with respect to the norm $\|\cdot\|_0$. With respect to the maximum norm the map $M_N(\cdot, p) : x \mapsto E_N p + Q_N LF(x)$ becomes a contraction for sufficiently large N (because the norm of the operator $Q_N L$ is bounded by $C\log(N)/N$, and F has a Lipschitz constant K with respect to $\|\cdot\|_0$ in $B^{0,1}_{6\delta}(x_0)$).

After the existence of the fixed point of $M_N(\cdot, p)$ is established in Section 5.3 the equivalence between the algebraic system $g(p) = 0$ and the periodic BVP $\dot{x}(t) = f(\Delta_t x)$ in the smaller ball $B^{0,1}_{\delta}(x_0)$ follows from the Splitting Lemma 2.4.

The smoothness (in the classical sense) of the maps X and g follows, colloquially speaking, from implicit differentiation of the fixed-point problem $x = E_N p + Q_N LF(x)$ with respect to p. Section 5.5 checks the uniform convergence of the difference quotient in detail, Section 5.6 uses the higher degrees of $EC^{j_{\max}}$ smoothness of f to prove higher degrees of smoothness for X and g. For proving higher-order smoothness one has to check only if the spectral radius of a linear operator is less than unity, but the inductive argument requires more elaborate notation than the first-order continuous differentiability.

3. Application to periodic orbits of autonomous FDEs — Hopf Bifurcation Theorem

Let us come back to the original problem, the parameter-dependent FDE (1) $\dot{x} = f(x_t, \mu)$, where $\mu \in \mathbb{R}^\nu$ is a system parameter and the functional $f : C^0(J;\mathbb{R}^n) \times \mathbb{R}^\nu \mapsto \mathbb{R}^n$ is defined for first arguments that exist on an arbitrary compact interval J. Periodic orbits are solutions

x of $\dot{x} = f(x_t, \mu)$ that are defined on $\mathbb{R}$ and satisfy $x(t) = x(t+T)$ for some $T > 0$ and all $t \in \mathbb{R}$.

Let x be a periodic function of period $T = 2\pi/\omega$. Then the function $y(s) = x(s/\omega)$ is a function of period 2π ($s \in \mathbb{T}$). This makes it useful to define the map

$$S : BC^0(\mathbb{R};\mathbb{R}^n) \times \mathbb{R} \mapsto BC^0(\mathbb{R};\mathbb{R}^n) \qquad [S(y,\omega)](s) = y(\omega s),$$

such that $S(y,\omega)(t) = x(t)$ for all $t \in \mathbb{R}$ (remember that $BC^0(\mathbb{R};\mathbb{R}^n)$ is the space of bounded continuous functions on the real line). Then $x \in C^1(\mathbb{R};\mathbb{R}^n)$ satisfies the differential equation

$$\dot{x}(t) = f(x_t, \mu) \tag{20}$$

on the real line and has period $2\pi/\omega$ if and only if $y = S(x, 1/\omega) \in C^1(\mathbb{T};\mathbb{R}^n)$ satisfies the differential equation

$$\dot{y}(s) = \frac{1}{\omega} f(S(\Delta_s y, \omega), \mu).$$

Let us define an extended differential equation

$$\dot{x}_{\mathrm{ext}}(s) = f_{\mathrm{ext}}(\Delta_s x_{\mathrm{ext}}), \tag{21}$$

where f_{ext} maps $C^0(\mathbb{T};\mathbb{R}^{n+1+\nu})$ into $\mathbb{R}^{n+1+\nu}$ and is defined by

$$f_{\mathrm{ext}} \begin{pmatrix} y \\ \omega \\ \mu \end{pmatrix} = \begin{bmatrix} f\left(S(y, R_0\omega), R_0\mu\right) / \operatorname{cut}(R_0\omega) \\ 0 \\ 0 \end{bmatrix}, \quad \text{where}$$

$$\operatorname{cut}(\omega) = \begin{cases} \omega & \text{if } \omega > \omega_{\mathrm{cutoff}} > 0 \\ \text{smooth, uniformly non-negative extension} & \text{for } \omega < \omega_{\mathrm{cutoff}} \end{cases}$$

for $y \in C^0(\mathbb{T};\mathbb{R}^n)$, $\omega \in C^0(\mathbb{T};\mathbb{R})$ and $\mu \in C^0(\mathbb{T};\mathbb{R}^\nu)$ (recall that R_0 takes the average of a function on $\mathbb{T}$). We have used in our definition that any functional f defined for $x \in C^0(J;\mathbb{R}^n)$ is also a functional on $C^0(\mathbb{T};\mathbb{R}^n)$ (periodic functions have a natural extension $x(t) = x(t_{\mathrm{mod}[-\pi,\pi)})$ if $t \in \mathbb{R}$ is arbitrary). The extended system has introduced the unknown ω and the system parameter μ as functions of time, and the additional differential equations $\dot{\omega} = 0$, $\dot{\mu} = 0$, which force the new functions to be constant for solutions of (21). We have also introduced a cut-off for ω close to zero to keep f_{ext} globally defined. The extended BVP (21) is in the form of periodic BVPs covered by the Equivalence Theorem 2.5. Thus, if f_{ext} is $EC^{j_{\max}}$ smooth then BVP (21) satisfies the assumptions of Theorem 2.5 in the vicinity of every periodic function $x_{0,\mathrm{ext}} \in C^{1,1}(\mathbb{T};\mathbb{R}^{n+\nu+1})$. Any solution $x_{\mathrm{ext}} = (y, \omega, \mu)$ that we find for (21) corresponds to a periodic solution $t \mapsto y(\omega t)$ of period $2\pi/R_0\omega$ at parameter $R_0\mu$ for (20) and vice versa, as long as $R_0\omega > \omega_{\mathrm{cutoff}}$. The condition of $EC^{j_{\max}}$ smoothness has to be checked only for the first n components of the function f_{ext} since its final $\nu + 1$ components are zero.

Application of the Equivalence Theorem 2.5 results in a system of algebraic equations that has $(n+\nu+1)(2N+1)$ variables and equations, where N is the positive integer proven

to exist in Theorem 2.5. Let us denote as $F=(F_y,F_\omega,F_\mu)$ the components of the right-hand side F_{ext} (given by $F_{\mathrm{ext}}(x_{\mathrm{ext}})(t)=f_{\mathrm{ext}}(\Delta_t x_{\mathrm{ext}})$), of which F_μ and F_ω are identically zero. Let $p=(p_y,p_\omega,p_\mu)$ be the $2N+1$ leading Fourier coefficients of y, ω and μ, respectively (these are the variables of the algebraic system constructed via Theorem 2.5), and $X(p)=(X_y(p),X_\omega(p),X_\mu(p))$ be the map from $R^{(n+\nu+1)(2N+1)}$ into $C^{j_{\max}}(\mathbb{T};\mathbb{R}^{n+\nu+1})$. Then several of the components of p can be eliminated as variables, and the equations for p resulting from Theorem 2.5 correspondingly simplified. Since F is identically zero in its last $\nu+1$ components we have

$$X_\omega(p)=E_N p_\omega, \qquad\qquad X_\mu(p)=E_N p_\mu.$$

Hence, the right-hand side

$$g(p)=R_N\left[P_0F(X(p))+Q_0\left(E_Np-P_NLF(X(p))\right)\right],$$

defined in Theorem 2.5, has $\nu+1$ components that are identical to zero (since $P_0F(X(p))=0$ for the equations $\dot\omega=0$ and $\dot\mu=0$). Furthermore, $g(p)=0$ contains the equations $R_NQ_0E_Np_\omega=0$ and $R_NQ_0E_Np_\mu=0$, which require that all Fourier coefficients (except the averages $R_0\omega$ and $R_0\mu$) of μ and ω are equal to zero. This means (unsurprisingly) that the algebraic system forces ω and μ to be constant. Thus, we can eliminate $R_NQ_0E_Np_\omega$ and $R_NQ_oE_Np_\mu$ (which are $2N(\nu+1)$ variables), replacing them by zero, and drop the corresponding equations. Since ω and μ must be constant, we can replace the arguments p_ω and p_μ of X by the scalar $R_0E_Np_\omega$ (which we re-name back to ω) and the vector $R_0E_Np_\mu\in\mathbb{R}^\nu$ (which we re-name back to μ). This leaves the first $n(2N+1)$ algebraic equations

$$0=R_N\left(P_0F_y(X_y(p_y,\omega,\mu),\omega,\mu)+Q_0\left[E_Np_y-P_NLF_y(X_y(p_y,\omega,\mu),\omega,\mu)\right]\right), \tag{22}$$

which depend smoothly (with degree $j_{\max}$) on the $n(2N+1)$ variables p_y and the parameters $\omega\in\mathbb{R}$ and $\mu\in\mathbb{R}^\nu$. Overall, (22) is a system of $n\times(2N+1)$ equations.

Rotational Invariance

The original nonlinearity F, defined by $[F(x)](t)=f(\Delta_t x)$ is equivariant with respect to time shift: $\Delta_tF(x)=F(\Delta_tx)$ for all $t\in\mathbb{T}$ and $x\in C^0(\mathbb{T};\mathbb{R}^n)$. Furthermore, Δ_t commutes with the following operations:

$$\Delta_tQ_NL=Q_NL\Delta_t \quad \text{(if } N\geq 0\text{) and} \qquad\qquad \Delta_tP_N=P_N\Delta_t.$$

This property gets passed on to the algebraic equation in the following sense: let us define the operation Δ_t for a vector p in $\mathbb{R}^{n(2N+1)}$, which we consider as a vector of Fourier coefficients of the function $E_Np\in C^0(\mathbb{T};\mathbb{R}^n)$, by

$$\Delta_tp=R_N\Delta_tE_Np.$$

With this definition Δ_t commutes with R_N and E_N. It is a group of rotation matrices: Δ_t is regular for all t, and $\Delta_{2k\pi}$ is the identity for all integers k. The definition of $X(p)$ as a

fixed point of $x \mapsto E_N p + Q_N LF(x)$ implies that $\Delta_t X(p) = X(\Delta_t p)$. From this it follows that the algebraic system of equations is also equivariant with respect to Δ_t. If we denote the right-hand-side of the overall system (22) by $G(p_y, \omega, \mu)$ then G satisfies

$$\Delta_t G(p_y, \omega, \mu) = G(\Delta_t p_y, \omega, \mu) \quad \text{for all } t \in \mathbb{T},\ p_y \in \mathbb{R}^{n(2N+1)},\ \omega > 0 \text{ and } \mu \in \mathbb{R}^{\nu}.$$

Application to Hopf bifurcation

One useful aspect of the Equivalence Theorem is that it provides an alternative approach to proving the Hopf Bifurcation Theorem for equations with state-dependent delays. The first proof that the Hopf bifurcation occurs as expected is due to Eichmann [5]. The reduction of periodic boundary-value problems to smooth algebraic equations reduces the Hopf bifurcation problem to an equivariant algebraic pitchfork bifurcation.

Let us consider the equation

$$0 = f(E_0 x_0, \mu) \tag{23}$$

where $f : C^0(J;\mathbb{R}^n) \times \mathbb{R} \mapsto \mathbb{R}^n$, $\mu \in \mathbb{R}$, J is a compact interval, $x_0 \in \mathbb{R}^n$, and the operator E_0 (as defined in (15) in Section 2) extends a constant to a function on $\mathbb{T}$ (and thus, on J). This means that (23) is a system of n algebraic equations for the $n+1$ variables (x_0, μ). The definition of EC^k smoothness does not cover functionals that depend on parameters. We avoid the introduction of a separate definition of EC^k smoothness for parameter-dependent functionals that distinguishes between parameters and functional arguments. We rather extend Definition 2.1:

Definition 3.1 (EC^k smoothness for parameter-dependent functionals)

> Let $J = [a, b]$ be a compact interval (or $J = \mathbb{T}$), and D be a product space of the form $D = C^{k_1}(J;\mathbb{R}^{m_1}) \times \ldots \times C^{k_\ell}(J;\mathbb{R}^{m_\ell})$ where $\ell \geq 1$, and $k_j \geq 0$ and $m_j \geq 1$ are integers. We say that $f : D \times \mathbb{R}^{\nu} \mapsto \mathbb{R}^n$ is EC^k smooth if the functional
>
> $$(x, y) \in D \times C^0(J;\mathbb{R}^{\nu}) \mapsto f(x, y(a)) \in \mathbb{R}^n \tag{24}$$
>
> is EC^k smooth (if $J = \mathbb{T}$ we use $a = -\pi$).

Requiring EC^k-smoothness of the parameter-dependent functional f in this sense, implies that the algebraic system $0 = f(E_0 x_0, \mu)$ is k times continuously differentiable. Let us assume that the algebraic system $0 = f(E_0 x_0, \mu)$ has a regular solution $x_0(\mu) \in \mathbb{R}^n$ for μ close to 0. Without loss of generality we can assume that $x_0(\mu) = 0$, otherwise, we introduce the new variable $x_{\text{new}} = x_{\text{old}} - E_0 x_0(\mu) \in C^0(J;\mathbb{R}^n)$. Hence, $f(0, \mu) = 0$ for all μ close to 0.

The EC^1 derivative of f in $(0, \mu)$ is a linear functional, mapping $C^0(J;\mathbb{R}^{n+\nu})$ into $\mathbb{R}^n$. Let us denote its first n components (the derivative with respect to the first argument x of f) by $a(\mu)$. The linear operator $a(\mu)$ can easily be complexified by defining $a(\mu)[x + iy] = a(\mu)[x] + ia(\mu)[y]$ for $x + iy \in C^0([-\tau, 0];\mathbb{C}^n)$. If f is EC^k smooth with $k \geq 2$ then the $n \times n$-matrix $K(\lambda, \mu)$ (called the *characteristic matrix*), defined by

$$K(\lambda, \mu)v = \lambda v - a(\mu)[v \exp(\lambda t)] \tag{25}$$

is analytic in its complex argument λ and $k-1$ times differentiable in its real argument μ (since the functions $t \mapsto v\exp(\lambda t)$ to which $a(\mu)$ is applied are all elements of $C^k(J;\mathbb{R}^n)$). The Hopf Bifurcation Theorem states the following:

Theorem 3.2 (Hopf bifurcation)

> Assume that f is EC^k smooth ($k \geq 2$) in the sense of Definition 3.1 and that the characteristic matrix $K(\lambda,\mu)$ satisfies the following conditions:
>
> 1. **(Imaginary eigenvalue)** there exists an $\omega_0 > 0$ such that $\det K(i\omega_0, 0) = 0$ and $i\omega_0$ is an isolated root of $\lambda \mapsto \det K(\lambda, 0)$. We denote the corresponding null vector by $v_1 = v_r + iv_i \in \mathbb{C}^n$ (scaling it such that $|v_r|^2 + |v_i|^2 = 1$).
> 2. **(Non-resonance)** $\det K(ik\omega, 0) \neq 0$ for all integers $k \neq \pm 1$.
> 3. **(Transversal crossing)** The local root curve $\mu \mapsto \lambda(\mu)$ of $\det K(\lambda,\mu)$ that corresponds to the isolated root $i\omega_0$ at $\mu = 0$ (that is, $\lambda(0) = i\omega_0$) has a non-vanishing derivative of its real part:
>
> $$0 \neq \frac{\partial}{\partial \mu} \operatorname{Re}\lambda(\mu)\bigg|_{\mu=0}.$$
>
> Then there exists a $k-1$ times differentiable curve
>
> $$\beta \in (-\varepsilon, \varepsilon) \mapsto (x, \omega, \mu) \in C^1(\mathbb{T};\mathbb{R}^n) \times \mathbb{R} \times \mathbb{R}$$
>
> such that for sufficiently small $\varepsilon > 0$ the following holds:
>
> 1. $x(\omega\cdot)$ (or $S(x,\omega)$) is a periodic orbit of $\dot{x}(t) = f(x_t, \mu)$ of period $2\pi/\omega$, that is, $x \in C^1(\mathbb{T};\mathbb{R}^n)$ and
>
> $$\dot{x}(t) = \frac{1}{\omega} f(S(\Delta_t x, \omega), \mu), \tag{26}$$
>
> 2. the first Fourier coefficients of x are equal to $(0, \beta)$, that is,
>
> $$\begin{aligned} 0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{Re}\left[v_1 \exp(it)\right]^T x(t)\,\mathrm{d}t, \quad \text{and} \\ \beta &= -\frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{Im}\left[v_1 \exp(it)\right]^T x(t)\,\mathrm{d}t, \end{aligned} \tag{27}$$
>
> 3. $x|_{\beta=0} = 0$, $\mu|_{\beta=0} = 0$ and $\omega|_{\beta=0} = \omega_0$, that is, the solution x, the system parameter μ and the frequency ω of x, which are differentiable functions of the amplitude β, are equal to $x = 0$, $\mu = 0$, $\omega = \omega_0$ for $\beta = 0$.

The statement is identical to the classical Hopf Bifurcation Theorem for ODEs in its assump-

tions and conclusions apart from the regularity assumption on f specific to FDEs. Note that the existence of the one-parameter family (parametrized in β) automatically implies the existence of a two-parameter family for $\beta \neq 0$ due to the rotational invariance: if x is a solution of (26) then $\Delta_s x$ is also a solution of (26) for every fixed $s \in \mathbb{T}$. Condition (27) fixes the time shift s of x such that x is orthogonal to $\operatorname{Re}[v_1 \exp(it)]$ using the $\mathbb{L}^2$ scalar product on $\mathbb{T}$.

The proof of the Hopf Bifurcation Theorem is a simple fact-checking exercise. We have to translate the assumptions on the derivative of $f : C^0(J;\mathbb{R}^n) \mapsto \mathbb{R}^n$ into properties of the right-hand side of the nonlinear algebraic system (22) near $(x, \omega, \mu) = (0, \omega_0, 0)$, and then apply algebraic bifurcation theory to the algebraic system. The only element of the proof that is specific to functional differential equations comes in at the linear level: the fact that the eigenvalue $i\omega_0$ is simple implies that the right nullvector $v_1 \in \mathbb{C}^n$ and any non-trivial left nullvector w_1 satisfy

$$w_1^H \left[\left.\frac{\partial}{\partial \lambda} K(\lambda, 0)\right|_{\lambda = i\omega_0} \right] v_1 \neq 0.$$

This is the generalization of the orthogonality condition $w_1^H v_1 \neq 0$, known from ordinary matrix eigenvalue problems, to exponential matrix eigenvalue problems of the type $K(\lambda, \mu) v = 0$. The proof of Theorem 3.2 is entirely based on the standard calculus arguments for branching of solutions to algebraic systems as can be found in textbooks [1]. The details of the proof can be found in Section 6.

The statements in Theorem 3.2 should be compared to two previous works considering the same situation: the branching of periodic orbits from an equilibrium losing its stability in FDEs with state-dependent delays. The PhD thesis of Eichmann (2006, [5]) proves the existence of the curve $\beta \mapsto (x(\beta), \mu(\beta), \omega(\beta))$ and that it is once continuously differentiable (assuming only EC^2 smoothness of f). Since $\mu'(0) = 0$ due to rotational symmetry (see proof in Section 6) this is not enough to determine if the non-trivial periodic solutions exist for $\mu > 0$ or for $\mu < 0$ for small β (the so-called *criticality* of the Hopf bifurcation, which is of interest in applications [15, 16]). Moreover, the non-resonance condition in [5] is slightly too strong, requiring that $i\omega_0$ is the only purely imaginary root of $\det K(\lambda, 0)$ (this assumption is different in the summary given in the review by Hartung *et al.* [12]; note that the publicly available version of [5] has a typo in the corresponding assumption L1), and only the pure-delay case (where the time interval J equals $[-\tau, 0]$) was considered. However, the techniques employed in [5], based on the Fredholm alternative, are likely to yield exactly the same result as stated in Theorem 3.2 if one assumes general EC^k smoothness with $k \geq 2$ (the formulation of EC^2 smoothness is already rather convoluted in [5]).

Hu and Wu [13] use S^1-degree theory [7, 18] to prove the existence of a branch of non-trivial periodic solutions near $(x, \mu, \omega) = (0, 0, \omega_0)$. This type of topological methods gives generally weaker results concerning the local uniqueness of branches of periodic orbits or their regularity, but they require only weaker assumptions ([13] still needs to assume EC^2 smoothness, though). Degree methods also give global existence results by placing restrictions on the number of branches that can occur.

4. Conclusion, applications and generalizations

Periodic boundary-value problems for functional differential equations (FDEs) are equivalent to systems of smooth algebraic equations if the functional f defining the right-hand side of the boundary-value problem satisfies natural smoothness assumptions. These assumptions are identical to those imposed in the review by Hartung *et al.* [12] and do not exclude FDEs with state-dependent delay. There are several immediate extensions of the results presented in this paper. The list below indicates some of them.

Further potential applications of the Equivalence Theorem 2.5

Theorem 3.2 on the Hopf bifurcation is not the central result of the paper, even though it is a moderate extension of the theorem proved in [5]. Rather, it is a demonstration of the use of the Equivalence Theorem 2.5. The main strength of the equivalence result stated in Theorem 2.5 is that it permits the straightforward application of continuation and Lyapunov-Schmidt reduction techniques to FDE problems involving periodic orbits of finite period, regardless if the delay is state dependent, or if the equation is of so-called mixed type (that is, positive and negative delays are present). A source of complexity, for example, in [5, 7, 10, 13, 18, 26], is that techniques such as Lyapunov-Schmidt reduction or S^1-degree theory had to be applied in Banach spaces. Theorem 2.5 removes the need for this, reducing the analysis of periodic orbits to root-finding in $\mathbb{R}^{n\times(2N+1)}$. For example, Humphries *et al.* [14] study periodic orbits in FDEs with two state-dependent delays numerically using DDE-Biftool [6], alluding to theoretical results about bifurcations of periodic orbits that have been proven only for constant delay. Fist of all, Humphries *et al.* [14] continue branches of periodic orbits. Theorem 2.5 makes clear when these branches as curves of points (x,ω,μ) in the extended space $C^0(\mathbb{T};\mathbb{R}^n)\times\mathbb{R}\times\mathbb{R}$ are smooth to arbitrary degree: the Jacobian of (22) with respect to (p_y,ω,μ) along the curve has to have full rank. Along these branches Humphries *et al.* [14] encounter degeneracies of the linearization and conjecture the existence of the corresponding bifurcations (backed up with numerical evidence) such as: fold bifurcations, period-doubling bifurcations, or the branching off of resonance surfaces (Arnol'd tongues) when resonant Floquet multipliers of the linearized equation cross the unit circle [20]. The Equivalence Theorem 2.5 provides a straightforward route to proofs that these scenarios occur as expected. Similarly, Theorem 2.5 will likely not only simplify the proofs about bifurcations of symmetric periodic orbits such as those of Wu [26], but also extend them to the case of FDEs with state-dependent delays. As long as one considers branching of periodic orbits with finite periods, the problem can be reduced locally (and often in every ball of finite size) to a finite-dimensional root-finding problem. This transfers also a list of results of symmetric bifurcation theory found in textbooks [8] to FDEs with state-dependent delays.

Globally valid algebraic system

The main result was formulated locally in the neighborhood of a given $x_0\in C^{1,1}(\mathbb{T};\mathbb{R}^n)$ and required only local Lipschitz continuity. The proof makes obvious that the domain of

definition for the map X, which maps between the function space and the finite-dimensional space is limited by the size of the neighborhood of x_0 for which one can find a uniform (EC) Lipschitz constant of the right-hand side F. In problems with delay the right-hand side is typically a combination of Nemytskii operators generated by smooth functions and the evaluation operator $\mathrm{ev} : C^0(\mathbb{T};\mathbb{R}^n) \times \mathbb{T} \mapsto \mathbb{R}^n$, given by $\mathrm{ev}(x,t) = x(t)$. These typically satisfy a (semi-)global Lipschitz condition (see also condition (Lb) in [12]): for all R there exists a constant K such that

$$|f(x) - f(y)| \le K\|x - y\|_0$$

for all x and y satisfying $\|x\|_1 \le R$ and $\|y\|_1 \le R$. Under this condition one can choose for any bounded ball an algebraic system that is equivalent to the periodic boundary-value problem in this bounded ball. For periodic orbits of autonomous systems this means that one can find an algebraic system such that all periodic orbits of amplitude less than R and of period and frequency at most R are given by the roots of the algebraic system.

Implicitly given delays

In practical applications the delay is sometimes given implicitly, for example, the position control problem considered in [23] and the cutting problem in [15, 16] contain a separate algebraic equation, which defines the delay implicitly. In simple cases these problems can be reduced to explicit differential equations using the standard Baumgarte reduction [2] for index-1 differential algebraic equations. For example, in the cutting problem the delay τ depends on the current position x via the implicit linear equation

$$\tau(t) = a - bx(t) - bx(t - \tau(t)), \tag{28}$$

which can be transformed into a differential equation by differentiation with respect to time:

$$\dot{\tau}(t) = \frac{-bv(t) - [\tau(t) - a + bx(t) + bx(t - \tau(t))]}{1 + bv(t - \tau(t))} \tag{29}$$

($v(t) = \dot{x}(t)$ is explicitly present as a variable in the cutting model, which is a second-order differential equation). The original model accompanied with the differential equation (29) instead of the algebraic equation (28) fits into the conditions of the Equivalence Theorem 2.5. The regularity statement of the Equivalence Theorem guarantees that the resulting periodic solutions have Lipschitz continuous derivatives with respect to time. This implies that the defect $d = \tau - (a - bx - bx(t - \tau))$ occurring in the algebraic condition (28) satisfies $\dot{d}(t) = -d(t)$ along solutions. Since the solutions are periodic the defect d is periodic, too, and, hence, d is identically zero. The denominator appearing in Equation (29) becomes zero exactly in those points in which the implicit condition (28) cannot be solved for the delay τ with a regular derivative.

The same argument can be applied to the position control problem as long as the object, at position x, does not hit the base at position $-w$ (the model contains the term $|x + w|$ in the right-hand side).

Neutral equations

The index reduction works only if the delay τ, which is itself a function of time, is not evaluated at different time points. For example, changing $bx(t-\tau(t))$ to $bx(t-\tau(t-1))$ on the right-hand-side of (28) would make the index reduction impossible. However, certain simple neutral equations permit a similar reduction directly on the function space level. Consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\Delta_t(x+g(x))\right]=f(\Delta_t x) \tag{30}$$

where the functional f satisfies the local EC Lipschitz condition, defined in Definition 2.2, in a neighborhood U of a point $x_0\in\mathbb{C}^{1,1}(\mathbb{T};\mathbb{R}^n)$, and $g:C^0(\mathbb{T};\mathbb{R}^n)\mapsto\mathbb{R}^n$ has a global (classical) Lipschitz constant less than unity (this excludes state-dependent delays in the essential part of the neutral equation). Then one can define the map $X_g(y)$ as the unique solution x of the fixed point problem

$$x(t)=y(t)-g(\Delta_t x)\quad\text{for } y \text{ near } y_0=x_0+g(x_0),$$

which reduces (30) to the equation

$$\dot{y}(t)=f(\Delta_t X_g(y))=f(X_g(\Delta_t y)). \tag{31}$$

Equation (31) satisfies the conditions of the Equivalence Theorem 2.5. One implication of this reduction is that periodic solutions of (30) are k times continuously differentiable if the functional f is EC^k smooth in the sense of Definition 2.1 and g is k times continuously differentiable as a map from $C^0(\mathbb{T};\mathbb{R}^n)$ into $\mathbb{R}^n$.

5. Proof of the Equivalence Theorem 2.5

Theorem 2.5 is proved in three steps. First, we establish the existence of a locally unique fixed point of the map $M_N(\cdot,p)$ using Banach's contraction mapping principle. This step requires only local EC Lipschitz continuity in the sense of Definition 2.2. In the second step we prove continuous differentiability of the map X and the right-hand side g of the algebraic system assuming that f is EC^1 smooth. In the final step we prove higher-order differentiability, assuming that f is EC^k smooth for degrees k up to $j_{\max}$.

5.1. Decay of Fourier coefficients for integrals and smooth functions

The following preparatory lemma states the well-known fact that, colloquially speaking, integrating a function makes its high-frequency Fourier coefficients smaller. In the fixed-point equation (18) of Theorem 2.5 the term $Q_N L$ occurs, and we need this term to be small for large N. Recall that Q_N removes the first N Fourier modes from a periodic function and Lx is the anti-derivative of x (after subtracting the average of x), see Equation (15) for the precise definitions.

Lemma 5.1 (Decay of Fourier coefficients of integrals)

> The norm of the linear operator $Q_N L$, mapping the space $C^j(\mathbb{T};\mathbb{R}^n)$ back into itself, is bounded by
> $$\|Q_N L\|_j \leq C\frac{\log N}{N}$$
> where C is a constant. The same holds in the Lipschitz norm (with the same constant C):
> $$\|Q_N L\|_{0,1} \leq C\frac{\log N}{N}.$$

Proof We find the norm $\|Q_N L\|_0$ first, and start out with the well-known estimate for interpolating trigonometric polynomials for continuous functions on $\mathbb{T}$. Let x be a continuous function on $\mathbb{T}$ with modulus of continuity ω. Then (see [17])

$$\|Q_N x\|_0 \leq C_0 \omega\left(\frac{2\pi}{N}\right)\log N$$

where C_0 is a constant that does not depend on x or N. A function $\omega : [0,\infty) \mapsto [0,\infty)$ is called a modulus of continuity of a continuous function x if

$$|x(t) - x(s)| \leq \omega(|t-s|).$$

holds for all s and $t \in \mathbb{T}$. For a function $x \in C^0(\mathbb{T};\mathbb{R}^n)$ the anti-derivative

$$[Lx](t) = \int_0^t x(s) - R_0 x \;\mathrm{d}s$$

has the Lipschitz constant $\|x\|_0 = \max\{|x(t)| : t \in \mathbb{T}\}$ such that a modulus of continuity for Lx is $\omega(h) = \|x\|_0 h$. Consequently,

$$\|Q_N Lx\|_0 \leq C_0 \frac{2\pi\|x\|_0}{N}\log N, \tag{32}$$

where C_0 does not depend on x or N. This proves the claim of the lemma for $j = 0$. For $x \in C^j(\mathbb{T};\mathbb{R}^n)$ we notice that all derivatives of x up to order j are continuous. Applying estimate (32) to each of the derivatives of x we get

$$\|Q_N Lx^{(l)}\|_0 \leq \frac{2\pi C_0}{N}\log N\,\|x^{(l)}\|_0 \quad \text{for } l = 0,\ldots j.$$

Consequently, the maximum of the left-hand sides over all $l \in \{0,\ldots,j\}$ must be less than the maximum of the right-hand sides:

$$\|Q_N Lx\|_j = \max_{l=0,\ldots,j}\|Q_N Lx^{(l)}\|_0 \leq \frac{2\pi C_0}{N}\log N \max_{l=0,\ldots,j}\|x^{(l)}\|_0 = 2\pi C_0\frac{\log N}{N}\|x\|_j,$$

which implies the desired estimate for $\|Q_N L\|_j$ using the constant $C = 2\pi C_0$.

The estimate of $Q_N L$ in the Lipschitz norm is a continuity argument. The operator $Q_N L$ is bounded (and, thus, continuous) on $C^{0,1}(\mathbb{T};\mathbb{R}^n)$. For every element y of $C^1(\mathbb{T};\mathbb{R}^n)$ (which is a dense subspace of $C^{0,1}(\mathbb{T};\mathbb{R}^n)$) the Lipschitz constant is identical to $\|y'\|_0 = \max_{t\in\mathbb{T}} |y'(t)|$, and, thus, $\|y\|_1 = \|y\|_{0,1}$. Let $x_n \in C^1(\mathbb{T};\mathbb{R}^n)$ be a sequence of continuously differentiable functions that converges to $x \in C^{0,1}(\mathbb{T};\mathbb{R}^n)$ in the $\|\cdot\|_{0,1}$-norm: $\|x_n - x\|_{0,1} \to 0$ for $n \to \infty$. Then

$$\|Q_N L x_n\|_{0,1} = \|Q_N L x_n\|_1 \le C\frac{\log N}{N}\|x_n\|_1 = C\frac{\log N}{N}\|x_n\|_{0,1}.$$

On both sides of the inequality the limit for $n \to \infty$ exists, resulting in the desired estimate for $\|Q_N L\|_{0,1}$. □

A direct consequence of Lemma 5.1 is that the Lipschitz norm of $Q_N x$, $\|Q_N x\|_{0,1}$, goes to zero for $N \to \infty$ for elements of $C^{1,1}(\mathbb{T};\mathbb{R}^n)$, so, for example, for a solution x of a periodic BVP:

$$\|Q_N x\|_{0,1} = \|Q_N L x'\|_{0,1} \le C\frac{\log N}{N}\|x'\|_{0,1} \le C\frac{\log N}{N}\|x\|_{1,1}. \tag{33}$$

5.2. Proof of Splitting Lemma 2.4

For any given integer $N \ge 0$ we have to show that the pair $(x,p) \in C^0(\mathbb{T};\mathbb{R}^n) \times \mathbb{R}^{n\times(2N+1)}$ satisfies

$$x(t) = x(0) + \int_0^t F(x)(s)\,\mathrm{d}s \quad \text{for all } t \in \mathbb{T}\text{, and} \tag{34}$$

$$p = R_N x, \quad \text{(or, equivalently, } E_N p = P_N x) \tag{35}$$

if and only if it satisfies the system

$$x = E_N p + Q_N L F(x), \tag{36}$$

$$0 = R_N\left[P_0 F(x) + Q_0\left(E_N p - P_N L F(x)\right)\right], \tag{37}$$

"⇒": Assume that $x \in C^0(\mathbb{T};\mathbb{R}^n)$ satisfies (34), and let $p = R_N x$. Subtracting equation (34) for $t = -\pi$ from (34) for $t = \pi$ implies that the average of $F(x)$ is zero. Thus, $R_0 F(x) = 0$ and $P_0 F(x) = 0$. Since $Ly = \int_0^t y(s) - R_0 y\,\mathrm{d}s$, the identity (34) implies (in combination with $R_0 F(x) = 0$)

$$x(t) = x(0) + [LF(x)](t). \tag{38}$$

Applying projection Q_N to this identity we obtain $Q_N x = Q_N L F(x)$. Adding (35) to this we obtain equation (36). Applying projection $P_N Q_0$ (which is the same as $Q_0 P_N$) to (38) we obtain $Q_0 P_N x = Q_0 P_N L F(x)$. Inserting $E_N p$ for $P_N x$ into this identity leads to $Q_0[E_N p - P_N L F(x)] = 0$. Since $P_0 F(x) = 0$, this in turn implies (37).

"⇐": Applying P_N to (36) implies $P_N x = E_N p$ (and $p = R_N x$) immediately. The expression inside the parentheses of R_N in equation (37) is a sum of two parts that each have to be zero (since they are both in the image of P_N on which R_N in injective). The projection Q_0 subtracts the average from its argument. Hence, $(x,p) \in C^0(\mathbb{T};\mathbb{R}^n) \times \mathbb{R}^{n\times(2N+1)}$ satisfies

(36)–(37) if and only if there exists a constant $c \in \mathbb{R}^n$ such that the triple (c, x, p) satisfies the system of equations consisting of (36) and

$$0 = R_0 F(x) \tag{39}$$

$$E_0 c = E_N p - P_N L F(x). \tag{40}$$

Note that E_0 maps the constant $c \in \mathbb{R}^n$ to a function that equals this constant for all $t \in \mathbb{T}$. In this system, (39) ensures that the average of $F(x)$ is zero. Equation (40) is an equation in the finite-dimensional space $\operatorname{rg} P_N$. Subtracting (40) from (36) gives

$$x = E_0 c + L F(x).$$

This equals (34), keeping in mind that $[Ly](t) = \int_0^t y(s) - R_0 y \, \mathrm{d}s$ ($[Ly](0) = 0$ for all $y \in C^0(\mathbb{T}; \mathbb{R}^n)$, hence, $x(0) = c$), and using $R_0 F(x) = 0$ (see equation (39)). □

5.3. Unique solvability of the fixed point problem (18)

Let x_0 be an element of $C^{1,1}(\mathbb{T}; \mathbb{R}^n)$, for example, a solution of the periodic boundary value problem $\dot{x}(t) = f(\Delta_t x) = F(x)(t)$. Consider a closed ball $B_\delta^{0,1}(x_0)$ of radius δ around x_0 in the Lipschitz norm:

$$B_\delta^{0,1}(x_0) = \{x \in C^{0,1}(\mathbb{T}; \mathbb{R}^n) : \|x - x_0\|_{0,1} \leq \delta\}.$$

The superscript "0, 1" indicates which norm is used to measure the distance from x_0 and that only elements of $C^{0,1}(\mathbb{T}; \mathbb{R}^n)$ are included.

Lemma 2.3 implies that F is Lipschitz continuous with respect to the $\|\cdot\|_0$-norm in $B_\delta^{0,1}(x_0)$ if we choose δ sufficiently small (thus, F is also called EC Lipschitz continuous in $B_\delta^{0,1}(x_0)$):

$$\|F(x) - F(y)\|_0 \leq K \|x - y\|_0 \tag{41}$$

for all x and y in $B_\delta^{0,1}(x_0)$ and a fixed $K > 0$. In any ball $B_\delta^{0,1}$, in which F is EC Lipschitz continuous, F is also bounded in the Lipschitz norm:

$$\|F(x)\|_{0,1} \leq R \quad \text{for all } x \in B_\delta^{0,1}(x_0). \tag{42}$$

See Lemma A.5 in Appendix A for the proof.

We can now formulate a lemma about the unique solvability of the fixed point problem

$$x = E_N p + Q_N L F(x).$$

This unique solvability and the Splitting Lemma 2.4 allow us to reduce the periodic BVP $\dot{x}(t) = f(\Delta_t x)$ to a system of algebraic equations. Remember that $E_N p$ takes a vector p of $2N + 1$ Fourier coefficients and maps it to the periodic function having these Fourier coefficients, $R_N x$ extracts the first $2N + 1$ Fourier coefficients from a periodic function x, $P_N x$ projects the periodic function x onto the space spanned by the basis $b_{-N}, \ldots, b_N$ and $Q_N = I - P_N$ sets the first Fourier modes of a function to zero. (P_N and Q_N are projections in the function space, and R_N and E_N map between the finite-dimensional subspace $\operatorname{rg} P_N$ and $\mathbb{R}^{n \times (2N+1)}$.)

Lemma 5.2 (Unique solvability of fixed point problem)

Let x_0 be in $C^{1,1}(\mathbb{T};\mathbb{R}^n)$, and let $\delta > 0$ be such that

$$\|F(x)\|_{0,1} \le R \quad \text{and} \quad \|F(x) - F(y)\|_0 \le K\|x - y\|_0 \tag{43}$$

for all x and $y \in B^{0,1}_{6\delta}(x_0)$ and for some constants $K > 0$ and $R > 0$ depending on δ. Then for any sufficiently large N the fixed point problem

$$x = E_N p + Q_N LF(x) \tag{44}$$

has a unique solution $x \in B^{0,1}_{6\delta}(x_0)$ for all vectors $p \in \mathbb{R}^{n\times(2N+1)}$ in the neighborhood U of $R_N x_0$ given by

$$U = \left\{p \in \mathbb{R}^{n\times(2N+1)} : \left\|E_N\left[p - R_N x_0\right]\right\|_{0,1} < 2\delta\right\}. \tag{45}$$

Moreover, if $x \in B^{0,1}_{\delta}(x_0)$ is continuously differentiable and satisfies $x' = F(x)$ then its projection $p = R_N x$ is in the neighborhood U, and x and p satisfy (44).

Note that U is an open set of $\mathbb{R}^{n\times(2N+1)}$ since E_N is an isomorphism between $\operatorname{rg} P_N$, equipped with the $\|\cdot\|_{0,1}$-norm, and $\mathbb{R}^{n\times(2N+1)}$. We have to prove the unique solvability of the fixed-point problem in a slightly larger ball (radius 6δ) and for a slightly larger range of parameters p (note the 2δ in (45)) in order to establish one-to-one correspondence in the ball of radius δ.

Proof The idea is, of course, that the function

$$M_N(\cdot, p) : x \mapsto E_N p + Q_N LF(x)$$

maps the closed ball $B^{0,1}_{6\delta}(x_0)$ back into itself and is uniformly contracting for suitably large N and vectors $p \in U$.

First, any closed ball $B^{0,1}_r(x_0)$ is closed (and, thus, forms a complete metric space) with respect to the $\|\cdot\|_0$-norm. This completeness is a simple continuity argument: let $y_n = x_0 + z_n$ be a Cauchy sequence in $B^{0,1}_r(x_0)$ with respect to the $\|\cdot\|_0$-norm. Then z_n converges to a continuous function z, and, since $\|z_n\|_0 \le \|z_n\|_{0,1} \le r$, for all n, the maximum norm of z is also bounded by r: $\|z\|_0 \le r$. We only have to show that the Lipschitz constant of z is bounded by r, too. Let $\varepsilon > 0$ be arbitrary and let $t \ne s$ be arbitrary in $\mathbb{T}$. We select some n such that $\|z - z_n\|_0 < \varepsilon|t - s|/2$. Then

$$\begin{aligned}|z(t) - z(s)| &\le |z(t) - z_n(t)| + |z_n(t) - z_n(s)| + |z_n(s) - z(s)| \\ &< \varepsilon|t - s| + r|t - s| \le (r + \varepsilon)|t - s|.\end{aligned}$$

Thus, the Lipschitz constant of z is less than $r + \varepsilon$ for arbitrary $\varepsilon > 0$. Hence, $\|z\|_{0,1} \le r$, completing the argument for completeness of $B^{0,1}_r(x_0)$ with respect to the $\|\cdot\|_0$-norm. This completeness implies that we can apply Banach's contraction mapping principle in a ball

$B_r^{0,1}(x_0)$, a ball of Lipschitz continuous functions, using the weaker maximum norm in the following.

We choose the radius r of the ball equal to 6δ (δ was chosen in the lemma such that the estimates (43) are true for the constants K and R), Thus, $B_{6\delta}^{0,1}(x_0)$ is the set to which we want to apply Banach's contraction mapping principle. To ensure that the map $M_N(\cdot,p)$ maps into $B_{6\delta}^{0,1}(x_0)$ for $p \in U$, and that $M_N(\cdot,p)$ is a contraction we pick N large enough. Specifically, we pick N such that

$$\begin{aligned} &\|Q_N x_0\|_{0,1} \leq 2\delta, \quad \|Q_N L\|_{0,1} \leq \frac{2\delta}{R}, \\ &\|Q_N L\|_0 \leq \frac{1}{2K}, \qquad C\frac{\log N}{N} < 1/\max\left\{1, \left(R + \|x_0\|_{1,1}\right)/\delta\right\}, \end{aligned} \tag{46}$$

where R and K are the bounds on F given in the conditions of the lemma, in Equation (43). We know that these bounds exist due to Lemma 2.3 (see Equation (41)) and Lemma A.5 (see Equation (42)). We know that choosing N according to (46) is possible from Lemma 5.1 and estimate (33) following Lemma 5.1.

Let us check first that $x \mapsto E_N p + Q_N LF(x)$ maps the closed ball $B_{6\delta}^{0,1}(x_0)$ back into itself:

$$\begin{aligned} &\left\|E_N p + Q_N LF(x) - x_0\right\|_{0,1} \leq \\ &\leq \left\|E_N\left[p - R_N x_0\right] - Q_N x_0 + Q_N LF(x)\right\|_{0,1} \\ &\leq \left\|E_N\left[p - R_N x_0\right]\right\|_{0,1} + \left\|Q_N x_0\right\|_{0,1} + \left\|Q_N L\right\|_{0,1}\|F(x)\|_{0,1} \\ &< 2\delta + 2\delta + \frac{2\delta}{R}R = 6\delta. \end{aligned}$$

Here we used the bounds (46) implied by our choice of N and the definition (45) of the set U of permitted p, and the bound on $\|F(x)\|_{0,1}$, which is determined in (43) by our choice of δ.

Second, let us check that $x \mapsto E_N p + Q_N LF(x)$ is a uniform contraction in $B_{6\delta}^{0,1}$ with respect to the $\|\cdot\|_0$-norm:

$$\left\|Q_N L\left[F(x) - F(y)\right]\right\|_0 \leq \left\|Q_N L\right\|_0 \left\|F(x) - F(y)\right\|_0 \leq \frac{1}{2K}K\|x-y\|_0 \leq \frac{1}{2}\|x-y\|_0.$$

Again, we exploited the bounds (46), implied by our choice of N, and the Lipschitz constant K of F determined in (43) by our choice of δ.

Since $B_{6\delta}^{0,1}(x_0)$ is complete with respect to the $\|\cdot\|$-norm Banach's contraction mapping principle implies that the fixed point problem (44) has a unique solution $x \in B_\delta^{0,1}(x_0)$ for $p \in U$.

Finally, let us check that for $x \in B_\delta^{0,1}(x_0) \cap C^1(\mathbb{T};\mathbb{R}^n)$ satisfying the periodic BVP $x' = F(x)$ the projection $p = R_N x$ is in U. For this we have to prove that if $\|x - x_0\|_{0,1} \leq \delta$ and

$x' = F(x)$ then $\|P_N(x - x_0)\|_{0,1} < 2\delta$. We can estimate $\|P_N(x - x_0)\|_{0,1}$ via

$$
\begin{aligned}
\|P_N(x-x_0)\|_{0,1} &\le \|(I-Q_N)(x-x_0)\|_{0,1} \le \|x-x_0\|_{0,1} + \|Q_N(x-x_0)\|_{0,1} && (47)\\
&\le \delta + C\frac{\log N}{N}\|x-x_0\|_{1,1} && (48)\\
&\le \delta + C\frac{\log N}{N}\max\{|x-x_0\|_{0,1}, \|x'-x_0'\|_{0,1}\} && (49)\\
&\le \delta + C\frac{\log N}{N}\max\{\delta, \|x'\|_{0,1} + \|x_0'\|_{0,1}\} && (50)\\
&= \delta + C\frac{\log N}{N}\max\{\delta, \|F(x)\|_{0,1} + \|x_0\|_{1,1}\} && (51)\\
&\le \delta + C\frac{\log N}{N}\max\{\delta, R + \|x_0\|_{1,1}\} < 2\delta. && (52)
\end{aligned}
$$

The inequality (47) follows from the definition of P_N and Q_N and the triangular inequality for the $\|\cdot\|_{0,1}$-norm. The step from (47) to (48) uses the estimate (33) for the norm $\|Q_N y\|_{0,1}$ for elements y of $C^{1,1}(\mathbb{T};\mathbb{R}^n)$. It also bounds $\|x - x_0\|_{0,1}$ by the radius δ of the ball. Step (49) splits up the $\|\cdot\|_{1,1}$ norm into its two parts which are estimated separately in the following steps. One part, $\|x - x_0\|_{0,1}$ is bounded by δ (the radius of the ball), the difference of the derivatives is bounded by a triangular inequality for its parts, $\|x'\|_{0,1}$ and $\|x_0'\|_{0,1}$ in (50). To get to (51) we use that x satisfies the BVP $x' = F(x)$. We also bound the norm of x_0' by $\|x_0\|_{1,1}$. Finally, in (52) we estimate the Lipschitz norm of $F(x)$, $\|F(x)\|_{0,1}$ by the bound R determined in (43) by our choice of δ. The right-hand side of (52) is (strictly) less than 2δ by our choice of N, see (46). □

5.4. Lipschitz continuity of the algebraic system

The Splitting Lemma 2.4 guarantees in combination with the unique existence of the fixed point of $M_N(\cdot, p)$, proven in Lemma 5.2, the equivalence between the periodic BVP $\dot{x}(t) = f(\Delta_t x)$ and the algebraic equation $g(p) = 0$ for x inside the ball $B^{0,1}_\delta(x_0)$, where g is given in (19) by

$$
\begin{aligned}
&g : p \in U \mapsto R_N\left[P_0 F(X(p)) + Q_0\left(E_N p - P_N L F(X(p))\right)\right] \in \mathbb{R}^{n\times(2N+1)}, \text{ where} && (53)\\
&X : p \in U \mapsto C^0(\mathbb{T};\mathbb{R}^n), \quad \text{and } X(p) \text{ is the fixed point of } M_N(\cdot,p) \text{ in } B^{0,1}_{6\delta}(x_0). && (54)
\end{aligned}
$$

The relation between $p \in U$ and $x \in B^{0,1}_\delta(x_0)$ is given via $p = R_N x$ and $x = X(p)$: if x satisfies the periodic BVP then $p = R_N x$ satisfies $g(p) = 0$, and, vice versa, if $p \in U$ satisfies $g(p) = 0$ then $x = X(p)$ satisfies the periodic BVP. The domain of definition, $D(X) = U$ is an open set, however the map X (and, thus, g) can be extended continuously to the boundary of U: $M_N(\cdot, p)$ maps into the closed ball $B^{0,1}_{6\delta}$ back into itself also for p on the boundary of U and it still has contraction rate $1/2$ with respect to the $\|\cdot\|_0$-norm.

The remainder of the section addresses the remaining open claim of the Equivalence Theorem 2.5, namely the regularity of the maps X and g. Using only local EC Lipschitz continuity (Definition 2.2) we can prove the Lipschitz continuity of g and X:

Lemma 5.3 (Regularity of X and algebraic system)

1. For all p in the neighborhood $U = D(X)$, defined in (45), the image $X(p)$ is in $C^{1,1}(\mathbb{T};\mathbb{R}^n)$ (that is, $X(p) \in C^1(\mathbb{T};\mathbb{R}^n)$ and its time derivative is Lipschitz continuous),
2. X is Lipschitz continuous with respect to the $\|\cdot\|_1$-norm for its images: there exists a constant C_N such that
$$\|X(p) - X(q)\|_1 \le C_N|p-q| \text{ for all } p \text{ and } q \text{ in } U,$$
3. the map $p \in U \mapsto \left[R_0F(X(p)), P_NLF(X(p))\right] \in \mathbb{R}^n \times \mathbb{R}^{n\times(2N+1)}$ is Lipschitz continuous in U.

Proof For a function $y \in \operatorname{rg} P_N$, differentiation is a bounded operator: $y' = D_N y$. The vector $R_N y$ of the first $2N+1$ Fourier coefficients of a function y and the vector $R_N[y']$ satisfy $R_N[y'] = \tilde{D}_N R_N y$ where $\tilde{D}_N$ is a matrix (independent of y). Hence, $y' = E_N\tilde{D}_N R_N y$ for all $y \in \operatorname{rg} P_N$ such that we can define $D_N = E_N\tilde{D}_N R_N$. Denote $X(p)$ as x. By definition of the map X, $x = E_N p + Q_N LF(x)$. The right-hand side of this fixed-point equation is differentiable with respect to time, giving

$$x' = D_N E_N p + Q_0F(x) - D_N P_N LF(x). \tag{55}$$

This guarantees that $x \in C^1(\mathbb{T};\mathbb{R}^n)$. Equation (42) ensures that $\|F(x)\|_{0,1} \le R$, which implies that the right-hand side of (55) is Lipschitz continuous in time. This in turn implies that x' is Lipschitz continuous in time (thus, $x \in C^{1,1}(\mathbb{T};\mathbb{R}^n)$), and

$$\|x'\|_{0,1} \le \|D_N E_N\|_{0,1}|p| + \|Q_0\|_{0,1}R + \|D_N P_N L\|_{0,1}R.$$

Representation (55) also implies point 2: let $x = X(p)$ and $y = X(q)$ be two functions in the image of X:

$$\|x'-y'\|_0 \le \|D_N E_N\|_0|p-q| + (\|Q_0\|_0 + \|D_N P_N L\|_0)K\|x-y\|_0, \tag{56}$$

where K was the EC Lipschitz constant of F in $B^{0,1}_{6\delta}(x_0)$. The difference $x-y$ in the $\|\cdot\|_0$-norm is bounded due to the contractivity of the right-hand side in fixed point problem (44) defining X (the $\|\cdot\|_0$-norm was the metric used to apply the contraction mapping principle):

$$\|x-y\|_0 \le \|E_N\|_0|p-q| + \|Q_N L[F(x) - F(y)\|_0 \le \|E_N\|_0|p-q| + \frac{1}{2}\|x-y\|_0.$$

Thus,

$$\|x-y\|_0 \le 2\|E_N\|_0|p-q|,$$

which, combined with (56), gives Lipschitz continuity of X as a map from U into $C^1(\mathbb{T};\mathbb{R}^n)$:

$$\|x' - y'\|_0 \le \left[\|D_N E_N\|_0 + (\|Q_0\|_0 + \|D_N P_N L\|_0) 2K \|E_N\|_0\right] |p-q| =: C_N |p-q|. \tag{57}$$

Point 3 is a direct consequence of the Lipschitz continuity of F with respect to $\|\cdot\|_0$-norm in $B^{0,1}_{6\delta}(x_0)$, the Lipschitz continuity of X on U in the $\|\cdot\|_0$-norm, and the fact that X maps into $B^{0,1}_{6\delta}(x_0)$. □

5.5. First-order differentiability of the algebraic system

Until now we have only used the EC Lipschitz continuity (in the sense of Definition 2.2) of the right-hand side F in the ball $B^{0,1}_{6\delta}(x_0)$ with respect to the $\|\cdot\|_0$-norm. We can expect that the right-hand side g of the algebraic system, defined in (53), is smooth only if we require more smoothness of the right-hand side f (which enters F in the algebraic system).

We first discuss first-order differentiability of the map X and the right-hand side g, defined in (53) and (54). For this we assume EC^1 smoothness of f as defined in Definition 2.1. For $x \in C^1(\mathbb{T};\mathbb{R}^n) \cap B^{0,1}_{6\delta}(x_0)$ the norm of $\partial^1 f(x,\cdot)$ as an element of $L(C^0(\mathbb{T};\mathbb{R}^n);\mathbb{R}^n)$ (the space of linear functionals mapping $C^0(\mathbb{T};\mathbb{R}^n)$ into $\mathbb{R}^n$) is less than or equal to K, the EC Lipschitz constant of F (and, hence, f) in $B^{0,1}_{6\delta}(x_0)$ assumed to exist in the conditions of Lemma 5.2.

Let us define the map

$$\partial^1 F : C^1(\mathbb{T};\mathbb{R}^n) \times C^0(\mathbb{T};\mathbb{R}^n) \mapsto C^0(\mathbb{T};\mathbb{R}^n), \qquad \left[\partial^1 F(v,w)\right](t) = \partial^1 f(\Delta_t v, \Delta_t w).$$

If $v \in C^1(\mathbb{T};\mathbb{R}^n)$ and $w \in C^{0,1}(\mathbb{T};\mathbb{R}^n)$ then the map $\partial^1 F$ defined above is indeed the derivative of F in v with respect to the deviation w (see Lemma A.6 in Appendix A):

$$\lim_{\substack{w \in C^{0,1}(\mathbb{T};\mathbb{R}^n) \\ \|w\|_{0,1} \to 0}} \frac{\|F(v+w) - F(v) - \partial^1 F(v,w)\|_0}{\|w\|_{0,1}} = 0. \tag{58}$$

Part of the definition of EC^1 smoothness for f is that the map $\partial^1 f$ is continuous in both arguments, $v \in C^1(\mathbb{T};\mathbb{R}^n)$ and $w \in C^0(\mathbb{T};\mathbb{R}^n)$. One can then apply Lemma A.4 to $\partial^1 f$ to conclude that the map $\partial^1 F$ (a composition of Δ_t and $\partial^1 f$) is continuous with respect to the $\|\cdot\|_0$-norm in its image space as a map of both arguments (in their respective norm), too. For $v \in B^{0,1}_{6\delta}(x_0)$ the norm of the linear map $\partial^1 F(v,\cdot)$ as an element of $L(C^0(\mathbb{T};\mathbb{R}^n); C^0(\mathbb{T};\mathbb{R}^n))$, the space of continuous linear functionals from $C^0(\mathbb{T};\mathbb{R}^n)$ back to itself, is bounded by the EC Lipschitz constant K of F in $B^{0,1}_{6\delta}(x_0)$.

The additional regularity assumption on f and its implications for F permit us to improve our statements about regularity of X and the algebraic system:

Lemma 5.4 (Continuous differentiability of X and the algebraic system)

> Assume that the right-hand side f is EC^1 smooth in the sense of Definition 2.1. Then the regularity statements about the map X, defined in (54), and the right-hand side of the algebraic system, defined in (53), can be extended:

1. $X(p)$ is in $C^2(\mathbb{T};\mathbb{R}^n)$ for all $p \in U = D(X)$, the domain of definition of X, and $p \mapsto X(p)$ is continuous with respect to the $\|\cdot\|_2$-norm for its images.
2. The map X, which maps U into $C^1(\mathbb{T};\mathbb{R}^n)$ according to Lemma 5.3, is continuously differentiable with respect to its argument p using the $\|\cdot\|_1$-norm for its images.
3. The map $p \in U \mapsto \left[R_0F(X(p)), P_N LF(X(p))\right] \in \mathbb{R}^n \times \mathbb{R}^{n\times(2N+1)}$ is continuously differentiable with respect to p.

Proof Let $p \in U = D(X) \subset \mathbb{R}^{n\times(2N+1)}$, where U is defined in (45), and let us denote $X(p)$ by x. Lemma 5.3 ensures already that x is in $C^{1,1}(\mathbb{T};\mathbb{R}^n)$. Lemma A.7 in Appendix A proves that $F(x) \in C^1(\mathbb{T};\mathbb{R}^n)$ for $x \in C^1(\mathbb{T};\mathbb{R}^n)$ (choosing $D = C^0(\mathbb{T};\mathbb{R}^n)$ and $k = 0$ in the assumptions of Lemma A.7). This implies the first statement, that $X(p) \in C^2(\mathbb{T};\mathbb{R}^n)$: since

$$X(p) = E_N p + Q_N LF(X(p)) \tag{59}$$

and $X(p) \in C^{1,1}(\mathbb{T};\mathbb{R}^n)$ (see Lemma 5.3), $F(X(p))$ is in $C^1(\mathbb{T};\mathbb{R}^n)$, and, thus, $LF(X(p))$ is in $C^2(\mathbb{T};\mathbb{R}^n)$. Hence, $X(p)$ is an element of $C^2(\mathbb{T};\mathbb{R}^n)$, too. Furthermore, Lemma A.7 states that F is continuous as a map from $C^1(\mathbb{T};\mathbb{R}^n)$ into $C^1(\mathbb{T};\mathbb{R}^n)$. Since X is continuous as a map from U into $C^1(\mathbb{T};\mathbb{R}^n)$ (in fact, it is Lipschitz continuous, see Lemma 5.3), the right-hand side of (59) in p is continuous with respect to the $\|\cdot\|_1$-norm. This proves the first point.

Concerning the second statement: again, let p_0 be in $U = D(X)$, and choose a small open neighborhood $U(p_0)$ which has a positive distance to the boundary of U. We will prove point two for all $p \in U(p_0)$. Let us choose an initial ε_0 sufficiently small such that $p + hq$ is still in U for $h \in (-\varepsilon_0, \varepsilon_0)$, all $p \in U(p_0)$, and all q with $|q| \le 1$. Let us introduce the difference quotient for $h \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$:

$$z(h,q,p) = \frac{1}{h}\left[X(p+hq) - X(p)\right]. \tag{60}$$

The maps z maps $\left[(-\varepsilon_0, \varepsilon_0) \setminus \{0\}\right] \times B_1(0) \times U(p_0) \subset \mathbb{R} \times \mathbb{R}^{n\times(2N+1)} \times \mathbb{R}^{n\times(2N+1)}$ into $C^1(\mathbb{T};\mathbb{R}^n)$. We first prove that z has a limit for $h \to 0$ in $C^1(\mathbb{T};\mathbb{R}^n)$, and that this limit is achieved uniformly for all $p \in U(p_0)$ and $|q| \le 1$. By definition of X, z satisfies the fixed point equation (dropping all arguments from z)

$$z = E_N q + Q_N L \frac{1}{h}\left[F(X(p) + hz) - F(X(p))\right] \tag{61}$$

for $h \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$. Let us introduce

$$\tilde{A}_1(p,z,h) = \begin{cases} \frac{1}{h}\left[F(X(p)+hz) - F(X(p))\right] & \text{if } h \neq 0 \\ \partial^1 F(X(p), z) & \text{if } h = 0, \end{cases} \tag{62}$$

which maps $U(p_0) \times C^{0,1}(\mathbb{T};\mathbb{R}^n) \times \mathbb{R}$ into $C^0(\mathbb{T};\mathbb{R}^n)$. The limit (58) implies that $\tilde{A}_1$ is continuous in all arguments (insert $v = x$, $w = hz$ into (58)). Using $\tilde{A}_1$ we extend the fixed point problem (61) to $h = 0$:

$$z = E_N q + Q_N L\tilde{A}_1(X(p), z, h). \tag{63}$$

The following intermediate lemma proves that the fixed point problem (63) has a unique solution:

Lemma 5.5 (Fixed point problem for linearization)

> There exists an $\varepsilon > 0$ and constants $C_0 > 0$ and $C_1 > 0$ such that the map
>
> $$\gamma : z \in C^{0,1}(\mathbb{T};\mathbb{R}^n) \mapsto E_N q + Q_N L\tilde{A}_1(X(p), z, h) \in C^{0,1}(\mathbb{T};\mathbb{R}^n),$$
>
> which depends on the additional parameters p, q and h, has a unique fixed point z_* in
>
> $$B = \{z \in C^{0,1}(\mathbb{T};\mathbb{R}^n) : \|z\|_0 \le C_0 \text{ and } \|z\|_{0,1} \le C_1\}$$
>
> for all $h \in (-\varepsilon, \varepsilon)$, all $p \in U(p_0) \subset U = D(X) \subset \mathbb{R}^{n\times(2N+1)}$ and all $q \in \mathbb{R}^{n(2N+1)}$ with $|q| < 1$. The fixed point z_* is an element of $C^1(\mathbb{T};\mathbb{R}^n)$ and depends continuously on h, p and q with respect to the $\|\cdot\|_1$-norm.

Note that the ε we have to choose in Lemma 5.5 is smaller than the initial ε_0 for which the difference quotient z is defined.

Proof of Lemma 5.5 First of all, since $\tilde{A}_1$ is continuous in all arguments, the map γ is continuous. Moreover, since $x' = (X(p))'$ and $x = X(p)$ depend continuously on p (see Lemma 5.3 and expression (55)), the map γ also depends continuously on the parameters p, q and h (that is, the expression $E_n q + Q_N L\tilde{A}_1(X(p), z, h)$, defining γ, depends continuously on z, p, q and h with respect to the $\|\cdot\|_{0,1}$-norm). We choose the constants $C_0 > 0$ and $C_1 > 0$ such that

$$C_0 \ge 2\|E_N\|_0 \tag{64}$$
$$C_1 \ge \|D_N E_N\|_0 + \left(\|Q_0\|_0 + \|D_N P_N L\|_0\right) K C_0, \tag{65}$$

where K is the Lipschitz constant of F with respect to the $\|\cdot\|_0$-norm in $B^{0,1}_{6\delta}$.

We choose $\varepsilon \le \varepsilon_0$ such that for all z satisfying $\|z\|_{0,1} \le C_1$ and all $p \in U(p_0)$ the function $X(p) + hz$ is in $B^{0,1}_{6\delta}(x_0)$ for all $h \in (-\varepsilon, \varepsilon)$. This implies that for any z_1 and z_2 satisfying $\|z_1\|_{0,1} \le C_1$ and $\|z_2\|_{0,1} \le C_1$ we have

$$\begin{aligned} \frac{1}{h}\|F(X(p) + hz_1) - F(X(p) + hz_2)\|_0 &\le K\|z_1 - z_2\|_0, \\ \|\partial^1 F(X(p), z_1 - z_2)\|_0 &\le K\|z_1 - z_2\|_0, \end{aligned} \tag{66}$$

where K was the Lipschitz constant for F in $B^{0,1}_{6\delta}(x_0)$, and, thus,

$$\|\gamma(z_1) - \gamma(z_2)\|_0 \le \frac{1}{2}\|z_1 - z_2\|_0 \tag{67}$$

for all $h \in (-\varepsilon, \varepsilon)$ by choice of N (N was such that $\|Q_N L\|_0 \le (2K)^{-1}$). This estimate for γ implies

$$\|\gamma(z)\|_0 \le \|E_N\|_0 + \frac{1}{2}\|z\|_0 \qquad \text{if } \|z\|_{0,1} \le C_1, \tag{68}$$

since $\gamma(0) = E_N q$ and $|q| \le 1$. Moreover, the two inequalities (66) imply that for $h \in (-\varepsilon, \varepsilon)$, $\|z\|_{0,1} \le C_1$ and $p \in U(p_0)$ the maximum norm of $\tilde{A}_1(p, z, h)$ is bounded by $K\|z\|_0$:

$$\|\tilde{A}_1(p, z, h)\|_0 \le K\|z\|_0 \tag{69}$$

The time derivative of $\gamma(z)$ exists and its $\|\cdot\|_0$-norm can be estimated by differentiating the expression $E_n q + Q_N L\tilde{A}_1(X(p), z, h)$, defining γ, with respect to time in the same manner as we obtained (56) (we insert (69) to bound $\|\tilde{A}_1(p, z, h)\|_0$):

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\gamma(z)\right\|_0 \le \|D_N E_N\|_0 + (\|Q_0\|_0 + \|D_N P_N L\|_0)K\|z\|_0. \tag{70}$$

The combination of the bounds (68) and (70) and the definition of the constants C_0 and C_1 guarantee that $\gamma(z)$ maps the set

$$B = \{z \in C^{0,1}(\mathbb{T}; \mathbb{R}^n) : \|z\|_0 \le C_0 \text{ and } \|z\|_{0,1} \le C_1\}$$

back into itself. The contraction estimate (67) for the $\|\cdot\|_0$-norm and the completeness of B with respect to the $\|\cdot\|_0$-norm make the contraction mapping principle applicable with a uniform contraction rate for all $p \in U(p_0)$, all $|q| \le 1$ and $h \in (-\varepsilon, \varepsilon)$. This ensures that the fixed point z_* depends continuously on p, $q \in \mathbb{R}^{n(2N+1)}$ and $h \in (-\varepsilon, \varepsilon)$ with respect to the $\|\cdot\|_0$-norm (since γ is continuous with respect to z, h, q and p).

The time derivative z_*' of z_* also exists and is continuous in p, q and h: we differentiate the fixed point equation (63) for z_* with respect to time (in the same way as done in (55)) to get

$$z_*' = D_N E_N q + Q_0\tilde{A}_1(X(p), z_*, h) - D_N P_N L\tilde{A}_1(X(p), z_*, h), \tag{71}$$

which is a continuous function in p, q and h with respect to the $\|\cdot\|_0$-norm (note that z_* depends on $p \in U(p_0)$, q and h). Thus, the fixed point z_* is in $C^1(\mathbb{T}; \mathbb{R}^n)$ and depends continuously on p, q and h with respect to the $\|\cdot\|_1$-norm. □

Proof of Lemma 5.4 continued As a consequence of Lemma 5.5 we may write the fixed point z_* of γ as a function of h, q and p: $z_*(h, q, p)$ maps $h \in (-\varepsilon, \varepsilon)$, q in the unit ball of $\mathbb{R}^{n\times(2N+1)}$ and $p \in U(p_0)$ continuously into $C^1(\mathbb{T}; \mathbb{R}^n)$. It is also identical to $z(h, q, p)$, defined in (60) as the directional difference quotient of X. Thus, the directional difference quotient $z(h, q, p)$ has a limit for $h \to 0$ in the $\|\cdot\|_1$-norm, and this limit equals $z_*(0, q, p)$. Moreover, this limit $z_*(0, q, p)$ depends continuously on p and q in the $\|\cdot\|_1$-norm (as proved in Lemma 5.5), and it is linear in q (since $\tilde{A}_1(p, z, 0)$ is linear in z). Thus, $z_*(0, q, p)$ is the Frechét derivative:

$$\lim_{q\to 0} \frac{\|X(p+q) - X(p) - z_*(0, q, p)\|_1}{|q|} = 0. \tag{72}$$

Consequently, the map $(p,q) \mapsto z_*(0,q,p) = \partial^1 X(p)q$ is continuous in the $\|\cdot\|_1$-norm as claimed in the lemma.

The third statement of Lemma 5.4 is a consequence of the second statement and the fact that the difference quotient of F has a limit in the $\|\cdot\|_0$-norm if it is taken between arguments in $C^1(\mathbb{T};\mathbb{R}^n)$ (see (58)). We split the difference quotients into two parts:

$$\frac{F(X(p+hq))-F(X(p))}{h} = \frac{F(X(p)+h\partial^1 X(p)q)-F(X(p))}{h} + \tag{73}$$

$$+\frac{F(X(p+hq))-F(X(p)+h\partial^1 X(p)q)}{h} \tag{74}$$

The right-hand side in (73) converges in the $\|\cdot\|_0$-norm to $\partial^1 F(X(p),\partial^1 X(p)q)$ for $h \to 0$, since $X(p)$ and $\partial^1 X(p)q$ are in $C^1(\mathbb{T};\mathbb{R}^n)$ because F is EC^1 continuous (see the second point of the lemma for the regularity of $\partial^1 X(p)q$ and Lemma 5.3 for the regularity of $X(p)$). For the term in (74) we can apply the local EC Lipschitz continuity (all arguments are in $B^{0,1}_{6\delta}(x_0)$ for $p \in U(p_0)$, $|q| \leq 1$ and $h \in (\varepsilon,\varepsilon)$) such that we get

$$\left\|\frac{F(X(p+hq))-F(X(p)+h\partial X(p)q)}{h}\right\|_0 \leq K\left\|\frac{X(p+hq)-X(p)}{h} - \partial X(p)q\right\|_0,$$

which converges to 0 for $h \to 0$ due to the second statement of the lemma (K is the EC Lipschitz constant of F in $B^{0,1}_{6\delta}(x_0)$). Consequently, we obtain from the limit of (73) for $h \to 0$ that the directional derivative of $F(X(p))$ in direction q is equal to $\partial^1 F(X(p),\partial^1 X(p)q)$, which is continuous with respect to p and q and linear in q. Thus,

$$\left[\frac{\partial}{\partial p}F(X(p))\right]q = \partial^1 F(X(p),\partial X(p)q), \tag{75}$$

and $p \mapsto F(X(p))$ is continuously differentiable with respect to p in the $\|\cdot\|_0$-norm. Note that we use the notation not enclosing q in the bracket in (75) to highlight that this is a classical derivative with respect to a finite-dimensional variable. The linear operators R_0 and $P_N L$ preserve the continuity (and the linearity in q) of (75). $\square$

Lemma 5.6 (Spectral radius of the linearized fixed point problem)

> For $x = X(p)$ (where $p \in U = D(X)$) consider the linear map
>
> $$M : z \mapsto Q_N L \partial^1 F(x,z).$$
>
> The spectral radius of M as a map from $C^0(\mathbb{T};\mathbb{R}^n)$ back into itself, or as a map from $C^1(\mathbb{T};\mathbb{R}^n)$ back into itself, is less or equal 1/2.

Proof Since M is compact as an element of $L(C^k(\mathbb{T};\mathbb{R}^n);C^k(\mathbb{T};\mathbb{R}^n))$ (the space of linear functionals from $C^k(\mathbb{T};\mathbb{R}^n)$ back to itself) for $k=0$ and $k=1$, the spectral radius is identical to the modulus of the maximal (in modulus) eigenvalue, which is of finite algebraic multiplicity if it is different from zero. An eigenvector z corresponding to this

maximal eigenvalue is an element of $C^1(\mathbb{T};\mathbb{R}^n)$ such that the spectral radius of M is the same for $k=0$ and $k=1$.

Since x and z are both in $C^1(\mathbb{T};\mathbb{R}^n)$ we have that

$$\partial^1 F(x)z = \lim_{h\to 0}\frac{1}{h}\left[F(x+hz)-F(x)\right] \tag{76}$$

For $x=X(p)$ where $p\in U$, and h sufficiently small the arguments of F, $x+hz$ and x, both lie inside $B^{0,1}_{6\delta}$ such that the EC Lipschitz constant K applies to the difference:

$$\frac{1}{h}\|F(x+hz)-F(x)\|_0 \le K\|z\|_0.$$

Since $\|Q_N L\|_0\le 1/(2K)$, (76) and (5.5) combine to

$$\|Mz\|_0 \le \frac{1}{2}\|z\|_0.$$

As z is an eigenvector corresponding to the largest eigenvalue, the spectral radius of M is less or equal $1/2$. □

Thus, the derivative $z=\partial X(p)q$ of X in p is the unique solution of the contractive linear fixed point problem in $C^1(\mathbb{T};\mathbb{R}^n)$

$$z = E_N q + Q_N L\partial^1 F(X(p),z). \tag{77}$$

5.6. Higher degrees of smoothness

We observe that $(x,y)=(X(p),\partial^1 X(p)q)$ satisfies the system of equations

$$\begin{aligned} x &= E_N p + Q_N L F(x)\\ y &= E_N q + Q_N L\partial^1 F(x,y). \end{aligned} \tag{78}$$

This has a similar structure to the original fixed point problem (44) but in dimension $n_1=2n$ with the variables (x,y) and parameters (p,q). Thus, we aim to apply a linear version of the arguments of Section 5.5 recursively, assuming that f is EC^k smooth as recursively defined in Definition 2.1. Throughout this section we assume that f is EC^k smooth for all degrees up to order $j_{\max}$.

For higher-order derivatives, we introduce the spaces D_j and the operators $\partial^j F$ for $j\ge 0$ recursively:

$$\begin{aligned} D_0 &= C^0(\mathbb{T};\mathbb{R}^n) & D_j &= D^1_{j-1}\times D_{j-1}\\ \partial^j F :&D_j \mapsto C^0(\mathbb{T};\mathbb{R}^n), & [\partial^j F(x)](t) &= \partial^j f(\Delta_t x). \end{aligned}$$

The spaces D_j are products of the type (9), and the argument x of $\partial^j F$ and $\partial^j f$ is in D_j, a product of 2^j spaces. We also recall that the notion of subspaces D^k_j of higher-oder ($k\ge 0$) differentiability for product spaces such as D_j was introduced in Section 2. For example,

$$\begin{aligned} &D^k_0 = C^k(\mathbb{T};\mathbb{R}^n),\\ &D^k_1 = D^{k+1}_0\times D^k_0 = C^{k+1}(\mathbb{T};\mathbb{R}^n)\times C^k(\mathbb{T};\mathbb{R}^n),\\ &D^k_2 = D^{k+1}_1\times D^k_1 = C^{k+2}(\mathbb{T};\mathbb{R}^n)\times C^{k+1}(\mathbb{T};\mathbb{R}^n)\times C^{k+1}(\mathbb{T};\mathbb{R}^n)\times C^k(\mathbb{T};\mathbb{R}^n), \text{ etc.,} \end{aligned}$$

all with their natural maximum norms. The maps $\partial^j F$ are all continuous and map indeed into $C^0(\mathbb{T};\mathbb{R}^n)$ due to the continuity of $\partial^j f$ and Δ_t (applying Lemma A.4 to D_j, $\partial^j F$ and $\partial^j f$). It is also clear from the definition that $\partial^{j+k}F = \partial^j[\partial^k F]$ if $j+k \le j_{\max}$. We will also use the notation $L(D^k_j;D^\ell_i)$ for the space of linear bounded functionals mapping from D^k_k into D^ℓ_i.

The following lemma is a consequence of the EC^k smoothness of f.

Lemma 5.7

> For $j+k \le j_{\max}$ the operator $\partial^j F$ is a continuous map from D^k_j into $C^k(\mathbb{T};\mathbb{R}^n)$.

Proof of Lemma We have to apply Lemma A.7 from Appendix A inductively over the order of differentiability (k). To start the induction for $k=0$ we can apply Lemma A.4 to D_j, $\partial^j F$ and $\partial^j f$. For the inductive step let us assume that for k we know that $\partial^j F : D^k_j \mapsto C^k(\mathbb{T};\mathbb{R}^n)$ is continuous for all $j \le j_{\max}-k$. Let us fix a $j \le j_{\max}-k-1$. We have to show that $\partial^j F$ maps D^{k+1}_j continuously into $C^{k+1}(\mathbb{T};\mathbb{R}^n)$. We know (by inductive assumption) that $\partial^j F$ maps D^k_j continuously into $C^k(\mathbb{T};\mathbb{R}^n)$ and that $\partial^{j+1}F$ maps $D^k_{j+1} = D^{k+1}_j \times D^k_j$ continuously into $C^k(\mathbb{T};\mathbb{R}^n)$. Thus, we can apply Lemma A.7 to $\partial^j F$ (this takes the place of the operator F in Lemma A.7) and $D = D^k_j$, obtaining that $\partial^j F : D^{k+1}_j \mapsto C^{k+1}(\mathbb{T};\mathbb{R}^n)$ is continuous. $\square$

An immediate consequence of Lemma 5.7 is that $X(p)$ and $\partial X(p)q$, as constructed in Section 5.5, are as smooth as the right-hand-side:

Lemma 5.8 (Smoothness of X and ∂X in time)

> Let f be $EC^{j_{\max}}$ smooth. For every $p \in U = D(X)$ and every $q \in R^{n(2N+1)}$ the functions $X(p)$ and $\partial X(p)q$ satisfy $X(p) \in C^{j_{\max}+1}(\mathbb{T};\mathbb{R}^n)$ and $\partial X(p)q \in C^{j_{\max}}(\mathbb{T};\mathbb{R}^n)$. Moreover, the maps
>
> $$p \mapsto X(p) \in C^{j_{\max}+1}(\mathbb{T};\mathbb{R}^n) \quad \text{and} \quad [p,q] \mapsto \partial X(p)q \in C^{j_{\max}}(\mathbb{T};\mathbb{R}^n)$$
>
> are continuous.

Proof The function $x = X(p)$ satisfies $x = E_N p + Q_N LF(x)$. Since F maps $D^k_0 = C^k(\mathbb{T};\mathbb{R}^n)$ back into itself continuously for all $k \le j_{\max}$, $Q_N L$ maps D^k_0 into D^{k+1}_0 continuously for all k, and $E_N p \in C^\infty(\mathbb{T};\mathbb{R}^n)$, the fixed point equation implies the following: if $x \in D^k_0$ then $F(x) \in D^k_0$, thus, $x = E_N p + Q_N LF(x) \in D^{k+1}_0$(for all $k \le j_{\max}$). Similarly, $z = E_N q + Q_N L\partial^1 F(x)z$, and $\partial^1 F$ maps D^k_1 into D^k_0 for all $k \le j_{\max}-1$. Thus, the fixed point equation implies: if $z \in D^k_0$ and $x \in D^{k+1}_0$ then $(x,z) \in D^k_1$, thus, $\partial^1 F(x,z) \in D^k_0$, thus, $z = E_N q + Q_N L\partial^1 F(x,z) \in D^{k+1}_0$ for all $k \le j_{\max}-1$. All of the above dependencies are continuous such that the continuous dependence on p and q in the norms of $D^{j_{\max}+1}_0$ and $D^{j_{\max}}_0$, respectively, follows. $\square$

We plan to find the derivatives of the map X inductively through fixed point equations

of the form (78). In order to set up the recursion we define inductively the operators F_j by

$$F_0(x) = F(x) \qquad \text{for } x \in D_0 \tag{79}$$

$$F_j \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} F_{j-1}(x) \\ \partial^1 F_{j-1}(x, y) \end{bmatrix}, \qquad \text{for } \begin{bmatrix} x \\ y \end{bmatrix} \in D_j = D^1_{j-1} \times D_j. \tag{80}$$

Note that F_j is always linear in its second argument, y, since $\partial^1 F_{j-1}$ is linear in its second argument. The operators F_j are combinations of derivatives of F. The plan is to study fixed-point problems of the type $x = E_N p + Q_N L F_j(x)$ (with $j = 1$ we obtain (78)). Before doing so, we establish which spaces the operators F_j map into:

Lemma 5.9 (Image of right-hand side)

> For $j + l + k \le j_{\max}$ the operator $\partial^l F_j$ maps D^k_{j+l} continuously into D^k_j. In particular, F_j maps D_j continuously back into itself.

Proof The statement of the lemma follows inductively from the definition of F_j and D^k_j. We apply Lemma 5.7 to start our induction over j (for $j = 0$ the statement is identical to Lemma 5.7). For the inductive step let us assume that we know that $\partial^l F_{j-1}$ maps D^k_{j+l-1} continuously into D^k_{j-1} for all k and l satisfying $l + k \le j_{\max} - j + 1$. By definition (80) of F_j the derivative $\partial^l F_j$ for $l \le j_{\max} - j$ is

$$\partial^l F_j \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \partial^l F_{j-1}(x) \\ \partial^{l+1} F_{j-1}(x, y) \end{bmatrix} \qquad \text{for } \begin{bmatrix} x \\ y \end{bmatrix} \in D_{l+j} = D^1_{l+j-1} \times D_{l+j-1}.$$

The first component, $\partial^l F_{j-1}$ maps D^{k+1}_{l+j-1} continuously into D^{k+1}_{j-1} for all k from 0 to $j_{\max} - l - j$ (this is the assumption of the inductive step when one shifts the index k down by 1). Similarly, $\partial^{l+1} F_{j-1}$ maps $D^{1+k}_{j+l-1} \times D^k_{j+l-1} = D^k_{j+l}$ continuously into D^k_{j-1} for all k from 0 to $j_{\max} - l - j$, again due to the assumption of the inductive step. Consequently, $\partial^l F_j$ maps $D^{k+1}_{j+l-1} \times D^k_{j+l-1} = D^k_{j+l}$ continuously into D^k_j for all k from 0 to $j_{\max} - l - j$, which is the statement we had to prove for the inductive step. □

Even though the map $x \in D^1_j \mapsto \partial^1 F_j(x, \cdot) \in L(D_j; D_j)$ is in general not continuous, the following map is:

Lemma 5.10 (Continuity in operator norm)

> For $j < j_{\max}$ the map $x \in D^1_j \mapsto Q_N L \partial^1 F_j(x, \cdot) \in L(D^1_j; D^1_j)$ is continuous with respect to $x \in D^1_j$.

Proof of Lemma 5.10 The EC^k smoothness of f (for $k \le j_{\max}$) implies that F_j is continuously differentiable (in the classical sense) as a map from D^1_j into D_j. Thus, the map $x \mapsto \partial^1 F_j(x, \cdot)$ as a map from D^1_j into $L(D^1_j; D_j)$ is continuous. Recall that the operator L involves taking the anti-derivative of its argument such that $L : D_j \mapsto D^1_j$. Since $Q_N L$ maps

D_j continuously into D^1_j, the map $x \mapsto Q_N L\partial^1 F_j(x,\cdot)$ is continuous as a map from D^1_j into $L(D^1_j; D^1_j)$. □

The following theorem provides continuous differentiability of order $j_{\max}$ for X and the map $p \mapsto F(X(p))$ if the right-hand side is EC^k smooth in the sense of Definition 2.1 for $k \le j_{\max}$:

Theorem 5.11 (Smoothness of algebraic system and X)

> Define $n_0 = n(2N+1)$ and $n_j = 2^j n_0$, and the maps
>
> $$X_0 : p \in U = D(X) \subseteq \mathbb{R}^{n_0} \mapsto X(p) \in D_0 \text{ and}$$
> $$Y_0 : p \in U = D(X) \subseteq \mathbb{R}^{n_0} \mapsto F(X(p)) \in D_0,$$
>
> and assume that $f : D_0 = C^0(\mathbb{T};\mathbb{R}^n) \mapsto \mathbb{R}^n$ is $EC^{j_{\max}}$ smooth. Then the following maps exist and are continuous for all j up to $j_{\max}$:
>
> $$X_j : [p,q] \in D(X_j) := D(X_{j-1}) \times \mathbb{R}^{n_{j-1}} \subseteq \mathbb{R}^{n_j} \mapsto [X_{j-1}(p), \partial X_{j-1}(p)q] \in D_j,$$
> $$Y_j : [p,q] \in D(X_j) \mapsto [Y_{j-1}(p), \partial Y_{j-1}(p)q] \in D_j.$$

The proof of Theorem 5.11 does not require the application of the contraction mapping principle for nonlinear maps. It uses only Lemma 5.6, Lemma 5.9 and Lemma 5.10 inductively.

Proof of Theorem 5.11 The main work is the proof of the existence and continuity of X_j, which we will do first.The assumption of the inductive step is comprised of the following two statements. We assume for j:

1. The map $(p_1,p_2) \in D(X_{j-1}) \times \mathbb{R}^{n_{j-1}} \mapsto X_j(p_1,p_2) \in D_j$ exists and is continuous. Moreover, the pair $(x_1,x_2) = X_j(p_1,p_2)$ satisfies
$$x_1 = E_N p_1 + Q_N L F_{j-1}(x_1) \tag{81}$$
$$x_2 = E_N p_2 + Q_N L\partial^1 F_{j-1}(x_1,x_2). \tag{82}$$

2. The linear map $z \mapsto Q_N L\partial^1 F_{j-1}(x_1,z)$ maps D^1_{j-1} back into itself and has spectral radius less or equal $1/2$.

Both statements of the assumption of the inductive step have been proven for $j=1$ in Lemma 5.4 and Lemma 5.6 . Let j be smaller than $j_{\max}$.

Regularity of $X_j(p)$

Let us first establish that the map

$$p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \mapsto x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = X_j \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{bmatrix} X_{j-1}(p_1) \\ \partial^1 X_{j-1}(p_1)p_2 \end{bmatrix}$$

does not only map continuously into D_j but into D_j^k for all $k \leq j_{\max} - j + 1$.

The argument is the same as in the proof of Lemma 5.8: the map F_j maps D_j^k continuously back into D_j^k for all $k \leq j_{\max} - j$. If $x \in D_j^k$ then $F_j(x) \in D_j^k$, thus, $x = E_N p + Q_N L F(x) \in D_j^{k+1}$ for all $k \leq j_{\max} - j$ (and the dependence on p is continuous because all dependencies are continuous).

Proof of existence and continuity of $\partial^1 X_j(p)q$

Let us use the notation $p = (p_1, p_2)$ and $x = (x_1, x_2) = X_j(p)$. Let $p_0 \in D(X_j)$ be arbitrary. We first show that $\partial^1 X_j(p)q$ exists for all p in a neighborhood $U(p_0)$ with positive distance to the boundary of $D(X_j)$. We can choose $\varepsilon > 0$ sufficiently small such that $p + hq \in D(X_j)$ for all $h \in (-\varepsilon, \varepsilon)$, all $q = (q_1, q_2)$ with $|q| < 1$ and all $p \in U(p_0)$. Consider the difference quotient

$$\frac{X_j(p+hq) - X_j(p)}{h} = \frac{1}{h} \begin{bmatrix} X_{j-1}(p_1 + hq_1) - X_{j-1}(p_1) \\ \partial^1 X_{j-1}(p_1 + hq_1)\,[p_2 + hq_2] - \partial^1 X_{j-1}(p_1)\,p_2 \end{bmatrix} =: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

By assumption of the inductive step, X_{j-1} is continuously differentiable such that the first row of this difference quotient has the form

$$z_1(h, p_1, q_1) = \frac{1}{h}\left[X_{j-1}(p_1 + hq_1) - X_{j-1}(p_1)\right] = \int_0^1 \partial^1 X_{j-1}(p_1 + hsq_1)\,q_1\,\mathrm{d}s \tag{83}$$

for $h \neq 0$. As established above $(p_1, q_1) \mapsto \partial^1 X_{j-1}(p_1)\,q_1 \in D_{j-1}^k$ is continuous for all $k \leq j_{\max} - j + 1$ such that

$$z_1(h, p_1, q_1) \in D_{j-1}^{j_{\max}-j+1} \subseteq D_{j-1}^2$$

($j_{\max} - j + 1 \geq 2$ since $j < j_{\max}$), and $z(h, p_1, q_1)$ depends continuously on its arguments, also when $h = 0$. Let us use the abbreviations

$$\begin{aligned} x_1(p_1) &= X_{j-1}(p_1), \\ x_2(p_1, p_2) &= \partial^1 X_{j-1}(p_1) p_2, \\ z_1(h, p_1, q_1) &= \frac{1}{h}\left[X_{j-1}(p_1 + hq_1) - X_{j-1}(p_1)\right] = \int_0^1 \partial^1 X_{j-1}(p_1 + hsq_1)\,q_1\,\mathrm{d}s \\ z_2(h, p_1, p_2, q_1, q_2) &= \frac{1}{h}\left[\partial^1 X_{j-1}(p_1 + hq_1)\,[p_2 + hq_2] - \partial^1 X_{j-1}(p_1)\,p_2\right] \quad \text{for } h \neq 0. \end{aligned}$$

With these notations we have $X_{j-1}(p_1 + hq_1) = x_1 + hz_1$ and, for non-zero h, $\partial^1 X_{j-1}(p_1 + hq_1)[p_2 + hq_2] = x_2 + hz_2$. The fixed-point equations (81) and (82) imply a fixed-point equation for the difference quotient z_2 for non-zero h:

$$\begin{aligned} z_2 &= E_N q_2 + \frac{1}{h} Q_N L \left[\partial^1 F_{j-1}(x_1 + hz_1, x_2 + hz_2) - \partial^1 F_{j-1}(x_1, x_2)\right] \\ &= E_N q_2 + \tilde{z}(p_1, p_2, q_1, h) + Q_N L \partial^1 F_{j-1}(x_1 + hz_1, z_2) \quad \text{where} \end{aligned} \tag{84}$$

$$\tilde{z}(p_1, p_2, q_1, h) = Q_N L \frac{\partial^1 F_{j-1}(x_1 + hz_1, x_2) - \partial^1 F_{j-1}(x_1, x_2)}{h}$$

The regularity of x_1, x_2 and z_1 is:

$$\begin{aligned} x_1 &\in D^{j_{\max}-j+2}_{j-1} \subseteq D^3_{j-1}, \\ x_2 &\in D^{j_{\max}-j+1}_{j-1} \subseteq D^2_{j-1} \quad \text{and} \\ z_1 &\in D^{j_{\max}-j+1}_{j-1} \subseteq D^2_{j-1}. \end{aligned} \tag{85}$$

We can apply the mean value theorem to the difference quotient appearing in $\tilde{z}$ since x_1 and z_1 are at least in D^2_{j-1} and x_2 is at least in D^1_{j-1} (see Lemma 5.7, and Lemma A.2 and Lemma A.6 in Appendix A):

$$\tilde{z}(p_1, p_2, q_1, h) = Q_N L \int_0^1 \partial^2 F_{j-1}(x_1 + shz_1, x_2, z_1, 0)\, \mathrm{d}s.$$

The map $(x_1, x_2, z_1, h) \mapsto \partial^2 F_{j-1}(x_1 + shz_1, x_2, z_1, 0)$ maps x_1, x_2, z_1 and h continuously into the space $D^{j_{\max}-j-1}_{j-1}$ (we see this by applying Lemma 5.7 to $\partial^2 F_{j-1}$, setting k in Lemma 5.7 to $j_{\max} - j - 1$). Thus, the quantity $\tilde{z}(p_1, p_2, q_1, h)$ is in $D^{j_{\max}-j}_{j-1} \subseteq D^1_{j-1}$ (since $j \leq j_{\max} - 1$). It depends continuously on p_1, p_2, q_1 and h in this space, and can be extended to $h = 0$ continuously (such that $\tilde{z}(p_1, p_2, q_1, 0) \in D^{j_{\max}-j}_{j-1}$, too).

Hence, (84) is a linear fixed-point problem for z_2 where the inhomogeneity is in $D^{j_{\max}-j}_{j-1}$ and depends continuously on (p, q, h). The linear map $M(h) : z_2 \mapsto Q_N L \partial^1 F_{j-1}(x_1 + hz_1) z_2$ in front of z_2 on the right-hand side of (84) depends continuously on h as an element of $L(D^1_{j-1}; D^1_{j-1})$ (see Lemma 5.10 and note that x_1 and z_1 are in D^1_{j-1}). Since the spectral radius of the map $M(0)$ (for $h = 0$) is less or equal than $1/2$ by assumption of our inductive step, the spectral radius of $M(h)$ is less than unity if we choose h sufficiently small. Thus, for all $p \in D(X_j)$ and $q \in \mathbb{R}^{n_j}$ and sufficiently small h, z_2 satisfies a contractive linear fixed point equation with an inhomogeneity in D^1_{j-1} and a contractive linear map that maps into D^1_{j-1} where all coefficients depend continuously on (h, p, q). Consequently, z_2 has a limit in D^1_{j-1} for $h \to 0$ that depends continuously on (p, q). For $h = 0$ the fixed point equation for (z_1, z_2) simplifies to

$$\begin{aligned} z_1 &= E_N q_1 + Q_N L \partial^1 F_{j-1}(x_1, z_1) \\ z_2 &= E_N q_2 + Q_N L \left[\partial^2 F_{j-1}(x_1, x_2, z_1, 0) + \partial^1 F_{j-1}(x_1, z_2) \right]. \end{aligned} \tag{86}$$

Both equations are linear in q and $z = (z_1, z_2)$. Consequently, $z(0, p, q)$, which is by definition the directional derivative of X_j in p in direction q, depends linearly on q and continuously on p and q. Consequently,

$$z(0, p, q) = \left[\frac{\partial}{\partial p} X_j(p) \right] q$$

is the Frechét derivative of X_j.

Collection to finish proof of statement 1 of inductive step

The functions $x = X_j(p)$ and $z = \partial^1 X_j(p)q$ satisfy

$$\begin{aligned} x =& E_N p + Q_N L F_j(x) && \text{by inductive assumption (81)–(82),} \\ z =& E_N q + Q_N L \partial^1 F_j(x,z) && \text{by (86) and definition of } F_j. \end{aligned} \tag{87}$$

The variable $x = X_j(p)$ depends continuously on p with respect to the norm of D_j^1 by the assumption of the inductive step and the step "Regularity of $X_j(p)$". The variable $z = \partial^1 X(p)q$ depends continuously on p and q as shown in the previous step, "Existence and continuity of $\partial^1 X(p)q$". Thus $(x,z) = (X_j(p), \partial^1 X(p)q) = X_{j+1}(p,q) \in D_j^1 \times D_j = D_{j+1}$ depends continuously on (p,q), and satisfies (81)–(82) for $j+1$ (which is identical to system (87)). This completes the proof of statement 1 of the inductive assumption for $j+1$.

Spectral radius of map $z \mapsto Q_N L \partial^1 F_j(x,z)$

The map $\partial^1 F_j$ maps D_{j+1} continuously into D_j (by Lemma 5.7). Thus, for fixed x the linear map $z \mapsto \partial^1 F_j(x,z)$ maps D_j continuously into D_j, and, hence, the map $M_j(x) : z \mapsto Q_N L \partial^1 F_j(x,z)$ maps D_j continuously into D_j^1, making $M_j(x)$ a compact linear operator. Thus, the spectral radius of $M_j(x)$ is determined by its largest eigenvalue (which has finite modulus and algebraic multiplicity if it is non-zero). Splitting $M_j(x)$ into its two components we get

$$M_j(x) : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} Q_N L \quad \partial^1 F_{j-1}(x_1, z_1) \\ Q_N L \left[\partial^2 F_{j-1}(x_1, x_2, z_1, 0) + \partial^1 F_{j-1}(x_1, z_2) \right] \end{bmatrix}$$

If $(\lambda, (z_1, z_2))$ is an eigenpair of $M_j(x)$ then the first row of the definition of $M_j(x)$ implies that, either (λ, z_1) is an eigenpair of $z_1 \mapsto Q_N L \partial^1 F_{j-1}(x_1, z_1)$, or $z_1 = 0$. If (λ, z_1) is an eigenpair of $z_1 \mapsto Q_N L \partial^1 F_{j-1}(x_1, z_1)$ then, by inductive assumption, $|\lambda| \le 1/2$. If $z_1 = 0$ then the term $\partial^2 F_{j-1}(x_1, x_2, z_1, 0)$ vanishes in the second row, such that (λ, z_2) is an eigenpair of $z_2 \mapsto Q_N L \partial^1 F_{j-1}(x_1, z_2)$. Thus, by inductive assumption, $|\lambda| \le 1/2$ in this case, too. Consequently, the spectral radius of $M_j(x)$ is also less or equal to $1/2$, which proves statement 2 of the inductive assumption for $j+1$.

Existence of Y_j We show inductively that $Y_j(p) = F_j(X_j(p))$. For $j = 1$ this statement was proven in Lemma 5.4. Let $j < j_{\max}$ and assume that $Y_j = F_j(X_j(p))$ for $p \in D(X_j)$. Since

$$X_j(p) = E_N p + Q_N L F_j(X_j(p))$$

and F_j maps D_j^1 into D_j^1, X_j is an element of D_j^1. Let $q \in \mathbb{R}^{n_j}$ be arbitrary, and let us denote $(x_1, x_2) = (X_j(p), \partial^1 X_j(p)q) = X_{j+1}(p,q)$. The component x_2 satisfies

$$x_2 = E_N q + Q_N L \partial^1 F_j(x_1, x_2)$$

such that x_2 is in D_j^1, too. Consequently,

$$\begin{aligned}\frac{Y_j(p+hq)-Y_j(p)}{h} &= \frac{F_j(X_j(p+hq))-F_j(X_j(p))}{h} \\ &= \frac{F_j(x_1+hx_2)-F_j(x_1)}{h} + \frac{F_j(X_j(p+hq))-F_j(x_1+hx_2)}{h}. \qquad (88)\end{aligned}$$

Since F_j is continuously differentiable for $x_1 \in D_j^1$ and deviations $hx_2 \in D_j^1$ the first quotient in the expression (88) converges to $\partial^1 F_j(x_1, x_2)$. Since F_j as a map from D_j^1 into D_j is locally Lipschitz continuous the second term in (88) can be bounded by

$$\begin{aligned}&\left\| \frac{F_j(X_j(p+hq)) - F_j(X_j(p) + h\partial^1 X_j(p)q)}{h} \right\|_{D_j} \leq \\ &\leq \; K_1 \left\| \frac{X_j(p+hq)-X_j(p)}{h} - \partial^1 X_j(p)q \right\|_{D_j},\end{aligned}$$

with some constant K_1, for sufficiently small h, which converges to zero for $h \to 0$ because X_j is differentiable. Consequently, the directional derivative of Y_j in p in direction q is $\partial^1 F_j(X_j(p))[\partial X_j(p)q]$, which is continuous in p and q and linear in q. Therefore, the Frechét derivative of Y_j exists and

$$\left[\frac{\partial}{\partial p} Y_j(p)\right] q = \partial^1 F_j(X_j(p), \partial X_j(p)q),$$

which implies by definition of F_j and X_j that $Y_{j+1} = F_{j+1}(X_{j+1}(p,q))$. □

We can refine the statement of Theorem 5.11 slightly by noting that $X_j : D(X_j) \mapsto D_j^1$ is continuous for all $j \leq j_{\max}$ (instead of $X_j : D(X_j) \mapsto D_j$). This follows from the continuity of $Y_j = F_j(X_j(p))$ as a map into D_j and the relation

$$X_j(p) = E_N p + Q_N L Y_j(p).$$

Theorem 5.11 completes the proof of the Equivalence Theorem 2.5. The refinement (that X_j maps into D_j^1) ensures that the image $X(p)$ is in $C^{j_{\max}+1}(\mathbb{T};\mathbb{R}^n)$, as claimed in Theorem 2.5

6. Proof of Hopf Bifurcation Theorem

First, we note that $x \mapsto S(x,\omega)^{-1} = x(\omega^{-1}\cdot)$ maps $C^k(\mathbb{T};\mathbb{R}^n)$ into a closed subspace of $C^k([-\tau,0];\mathbb{R}^n)$, if we extend functions x on $\mathbb{T}$ to the whole real line by setting $x(t) = x(t_{\mathrm{mod}[-\pi,\pi)})$. This implies that, if the functional $f : C^0([-\tau,0];\mathbb{R}^n) \times \mathbb{R} \mapsto \mathbb{R}^n$ is EC^k smooth then the functional

$$(x,\mu,\omega) \in C^0(\mathbb{T};\mathbb{R}^n) \times \mathbb{R}^2 \mapsto \frac{1}{\omega} f(S(x,\omega),\mu) \in \mathbb{R}^n$$

is EC^k smooth, too, such that we can reduce the problem of finding periodic orbits of frequency ω to the algebraic system (22). The right-hand side F_y in (22) is defined by

$$\left[F_y(x,\omega,\mu)\right](t) = \frac{1}{\omega} f(S(\Delta_t x,\omega),\mu).$$

Let us choose the periodic orbit $x_0 = (x,\omega,\mu)$ with $x = 0$, $\omega = \omega_0$, $\mu = 0$ as the solution in the neighborhood of which we construct the equivalent algebraic system. We choose the number N of Fourier modes and the size δ of the neighborhood $B_\delta^{0,1}(x_0)$ in $C^{0,1}(\mathbb{T};\mathbb{R}^{n+2})$ such that the conditions of Theorem 2.5 are satisfied in $B_\delta^{0,1}(x_0)$. The full algebraic system (22) then reads (after multiplication with ω and mapping it onto the space $\operatorname{rg} P_N$ from $\mathbb{R}^{n(2N+1)}$ by applying R_N^{-1})

$$0 = P_0 F_y(X_y(p,\omega,\mu),\omega,\mu) + \omega Q_0 P_N E_N p - Q_0 P_N L F_y(X_y(p,\omega,\mu),\omega,\mu) \tag{89}$$

The variables are $p \in \mathbb{R}^{n(2N+1)}$ (which was called p_y in (22)), μ and ω. We know from Theorem 2.5 that

$$\begin{aligned} Y_y &: (p,\omega,\mu) \in \mathbb{R}^{n(2N+1)} \times \mathbb{R} \times \mathbb{R} \mapsto F(X_y(p,\omega,\mu),\omega),\mu) \in C^0(\mathbb{T};\mathbb{R}^n), \\ X_y &: (p,\mu,\omega) \in \mathbb{R}^{n(2N+1)} \times \mathbb{R} \times \mathbb{R} \mapsto X_y(p,\omega,\mu) \in C^0(\mathbb{T};\mathbb{R}^n) \end{aligned}$$

are k times differentiable, and note that

$$F_y(X_y(0,\omega,\mu),\omega),\mu) = 0 \tag{90}$$

for all $\omega \approx \omega_0$ and $\mu \approx 0$ (because $x_0 = (0,\omega,\mu)$ is a solution). The derivative of the right-hand side F_y in $x = 0$, $\omega \approx \omega_0$ and $\mu \approx 0$ with respect to x is $A(\omega,\mu)x$, defined by

$$\left[A(\omega,\mu)x\right](t) = a(\mu)\left[x(t+\omega\cdot)\right],$$

where $a(\mu)$ is the same linear functional as used in the definition of the characteristic matrix $K(\lambda,\mu)$ in (25) (the derivatives of F with respect to ω and μ are zero due to (90)). We observe that $A(\omega,\mu)$ commutes with P_j and Q_j for all $j \geq 0$.

Let us now determine the linearization of $X_y(p,\omega,\mu)$ in $(p,\omega,\mu) = (0,\omega,\mu)$. Due to (90) $X_y(0,\omega,\mu)$ is equal to zero for all $\omega \approx \omega_0$ and $\mu \approx 0$: since 0 is a solution to the periodic BVP and $P_N 0 = 0$, the zero solution must also be equal to $X_y(0,\omega,\mu)$. Thus, we have

$$0 = \left.\frac{\partial}{\partial \omega} X_y(p,\omega,\mu)\right|_{p=0} \quad \text{and} \quad 0 = \left.\frac{\partial}{\partial \mu} X_y(p,\omega,\mu)\right|_{p=0}.$$

Moreover, the fixed point equation (77) defining $z = [\partial X_y/\partial p](p,\omega,\mu)q$, evaluated in $(p,\omega,\mu) = (0,\omega,\mu)$ reads

$$z = E_N q + Q_N L A(\mu,\omega) z = E_N q + Q_N L A(\mu,\omega) Q_N z, \tag{91}$$

exploiting that $Q_N L = Q_N L Q_N$ and $Q_N A(\omega,\mu) = A(\omega,\mu) Q_N$. In the neighborhood $B_\delta^{0,1}(x_0)$ the spectral radius of $Q_N L A(\mu,\omega)$ is less than unity (see Lemma 5.6). Application of Q_N

to (91) gives $Q_N z = Q_N LA(\mu,\omega)Q_N z$. Since the spectral radius of $Q_N LA(\mu,\omega)$ is less than unity this implies that $Q_N z = 0$, and, thus

$$\left[\frac{\partial}{\partial p}X_y(p,\omega,\mu)\Big|_{p=0}\right] q = E_N q.$$

Consequently, the linearization of the algebraic system (89) in $(0,\omega,\mu)$ with respect to the first variable is

$$0 = P_0 A(\omega,\mu)E_N p + \omega Q_0 P_N E_N p - Q_0 P_N LA(\omega,\mu)E_N p \tag{92}$$

for all $\omega \approx \omega_0$ and $\mu \approx 0$ (also using p for the argument of the linearization in (92)). We observe that the linear system (92) decouples into equations for

$$\begin{aligned}
&y_0 = P_o E_N p = E_0 p = p_0 && \text{(the average of } E_N p\text{)},\\
&y_1 = Q_0 E_1 p = p_{-1}\sin t + p_1 \cos t && \text{(the first Fourier component of } E_N p\text{)},\\
&y_j = Q_{j-1}E_j p = p_{-j}\sin(jt) + p_j\cos(jt) && \text{(the } j\text{-th Fourier component of } E_N p,\\
& && 2 \le j \le N),
\end{aligned}$$

where we denote the components of p by $p_j \in \mathbb{R}^n$ ($j = -N \dots N$). This decoupling is achieved by pre-multiplication of (92) with P_0 and $Q_{j-1}P_j$ for $j = 1 \dots N$:

$$\begin{aligned}
P_0 \cdot (92):& \qquad 0 = A(\omega,\mu)y_0 = a(\mu)p_0 && \quad (93)\\
Q_0 P_1 \cdot (92):& \qquad 0 = \omega y_1 - Q_0 LA(\omega,\mu)y_1 && \quad (94)\\
Q_{j-1}P_j \cdot (92):& \qquad 0 = \omega y_j - Q_0 LA(\omega,\mu)y_j \qquad \text{for } j = 2\dots N. && \quad (95)
\end{aligned}$$

Inserting the definition of y_j into the equations (94) and (95) gives for $j \ge 1$

$$\begin{aligned}
0 =& \omega\left[p_{-j}\sin(jt) + p_j\cos(jt)\right] - Q_0\int_0^t a(\mu)[p_j\sin(js + j\omega\cdot) + p_j\cos(js + j\omega\cdot)]\,\mathrm{d}s\\
=& \omega\left[p_{-j}\sin(jt) + p_j\cos(jt)\right] - \frac{1}{j}\sin(jt)a(\mu)[p_{-j}\sin(j\omega\cdot) + p_j\cos(j\omega\cdot)]\\
&- \frac{1}{j}\cos(jt)a(\mu)[-p_{-j}\cos(j\omega\cdot) + p_j\sin(j\omega\cdot)].
\end{aligned}$$

These equations are satisfied if and only if the coefficients in front of $\sin(jt)$ and $\cos(jt)$ are zero. The resulting system of equations reads in complex notation (splitting up again into the cases $j = 1$ and $j > 1$)

$$i\omega u_1 - a(\mu)\left[u_1\exp(i\omega s)\right] = K(i\omega,\mu)u_1 = 0, \tag{96}$$

$$ij\omega u_j - a(\mu)\left[u_j\exp(ij\omega s)\right] = K(ij\omega,\mu)u_j = 0 \quad (2 \le j \le N), \tag{97}$$

that is, $u_j = p_{-j} + ip_j \in \mathbb{C}^n$ is a solution of (96) (or (97), respectively) if and only if $y_j = p_{-j}\sin(jt) + p_j\cos(jt)$ is a solution of (94) (or (95), respectively).

The non-resonance assumption of the theorem guarantees that equation (93) is a regular linear system for p_0, and that (97) is a regular linear algebraic system for p_{-j} and p_j ($j \geq 2$) at $\mu = 0$ and $\omega = \omega_0$ (and, hence, for all ω and μ near-by). The condition on the simplicity of the eigenvalue $i\omega_0$ of K ensures that equation (96) (and, thus, (94)) has a one-dimensional (in complex notation) subspace of solutions for $\omega = \omega_0$ and $\mu = 0$, spanned by the nullvector v_1 of $K(i\omega, 0)$. Let us denote the adjoint nullvector of $K(i\omega_0, 0)$ by $w_1 \in \mathbb{C}^n$ (again, using complex notation, $w_1^H K(i\omega_0, 0) = 0$). Since $i\omega_0$ is simple, the relationship

$$w_1^H \frac{\partial K}{\partial \lambda}(i\omega, 0)\, v_1 \neq 0$$

holds, which implies that we can choose $w_1 \in \mathbb{C}^n$ without loss of generality such that

$$w_1^H \frac{\partial K}{\partial \lambda}(i\omega, 0)\, v_1 = 1.$$

With this convention we observe that

$$w_1^H \frac{\partial K}{\partial \mu}(i\omega, 0)\, v_1 = -\left.\frac{\partial \lambda}{\partial \mu}\right|_{\mu=0} =: c_\mu \in \mathbb{C}, \text{ and} \qquad w_1^H \frac{\partial}{\partial \omega} K(i\omega, 0)\, v_1 = i \in \mathbb{C} \tag{98}$$

where $\operatorname{Re} c_\mu \neq 0$ by the transversal crossing assumption of the theorem. In complex notation any scalar multiple of the nullvector $v_1 = v_r + i v_i$ is also a nullvector. Thus, the complex scalar factor $\alpha + i\beta$ in front of v_1 makes up two components of the variable p (in real notation): in short, p solves the linearized algebraic system (92) if and only if all p_j with $|j| \neq 1$ are zero and $p_{-1} \sin t + p_1 \cos t = \operatorname{Re}\left[(\alpha + i\beta) v_1 \exp(it)\right]$ for some $\alpha, \beta \in \mathbb{R}$, that is,

$$\begin{bmatrix} p_{-1} \\ p_1 \end{bmatrix} = \alpha \begin{bmatrix} -v_i \\ v_r \end{bmatrix} + \beta \begin{bmatrix} -v_r \\ -v_i \end{bmatrix} =: \alpha b_r + \beta b_i. \tag{99}$$

Let us collect the statements so far and introduce coordinates. We collect all components p_j with $|j| \neq 1$ and the orthogonal complement in $\mathbb{R}^{2n}$ of the space spanned by $\{b_1, b_2\}$ into a single variable q (of real dimension $n_q = n(2N-1) + 2(n-1)$). Then a set of coordinates for p are the variables

$$(\alpha, \beta) =: r \in \mathbb{R}^2, \quad \text{and} \quad q \in \mathbb{R}^{n_q}.$$

We split up the full algebraic system of equations (89) in the same way as we did for the linearized problem, by pre-multiplication with P_0 and $Q_{j-1}P_j$ for $j = 1 \ldots N$:

$$P_0 \cdot (89): \qquad 0 = P_0 F(X_y(p, \omega, \mu), \omega, \mu) \tag{100}$$

$$Q_0 P_1 \cdot (89): \qquad 0 = \omega Q_0 E_1 p - Q_0 P_1 L F(X_y(p, \omega, \mu), \omega, \mu) \tag{101}$$

$$Q_{j-1} P_j \cdot (89): \qquad 0 = \omega Q_{j-1} E_j p - Q_{j-1} P_j L F(X_y(p, \omega, \mu), \omega, \mu). \tag{102}$$

We split equation (101) further using w_1^H and its orthogonal complement, the projection $w_1^\perp = I - w_1 w_1^H / (w_1^H w_1)$. This gives rise to a splitting into two real equations ($w_1^H \cdot$(101)) and $2(n-1)$ real equations ($w_1^\perp \cdot$(101)). Collecting $w_1^\perp \cdot$(101) and the equations (100) and

([102](#)) into a subsystem of $n(2N-1)+2(n-1)=n_q$ equations the full algebraic system (89) in the coordinates (r,q) has the form

$$0=\begin{bmatrix} M_{rr}(r,q,\omega,\mu) & M_{rq}(r,q,\omega,\mu) \\ M_{qr}(r,q,\omega,\mu) & M_{qq}(r,q,\omega,\mu) \end{bmatrix}\begin{bmatrix} r \\ q \end{bmatrix}. \tag{103}$$

By our choice of coordinates the matrices $M_{rr}\in\mathbb{R}^{2\times 2}$, $M_{rq}\in\mathbb{R}^{2\times n_q}$ and $M_{qr}\in\mathbb{R}^{n_q\times 2}$ are identically zero in $r=0$, $q=0$, $\mu=0$, $\omega=i\omega_0$ such that the system matrix has the form

$$\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} & \begin{matrix} M_{qq}(0,0,i\omega,0) \\ \text{(regular)} \end{matrix} \end{bmatrix}$$

$(r,q,\mu,\omega)=(0,0,0,\omega_0)$. Thus, we can perform a Lyapunov-Schmidt reduction: we eliminate q by solving the n_q lower equations for q, obtaining a graph $q(r,\omega,\mu)\,r$ locally in a neighborhood of $(r,q,\mu,\omega)=(0,0,0,\omega_0)$. This graph respects rotational invariance: $q(\Delta_s r,\omega,\mu)\,\Delta_s r=\Delta_s[q(r,\omega,\mu)\,r]$. Note that the application of Δ_s to $r=(\alpha,\beta)$ corresponds to the rotation of r by angle s (the same as the multiplication $\exp(is)(\alpha+i\beta)$). The Lyapunov-Schmidt reduction of (103), replacing q by the graph $q(r,\omega,\mu)\,r$, then reads

$$0=M_{rr}(r,q(r,\omega,\mu)\,r,\omega,\mu)\,r=:M_r(r,\omega,\mu)\,r, \tag{104}$$

where M_r is still rotationally symmetric in r: $M_r(\Delta_s r,\omega,\mu)\,\Delta_s r=\Delta_s M_r(r,\omega,\mu)\,r$. Equation (98) in real notation implies that

$$\begin{aligned} \frac{\partial M_r}{\partial\omega}(0,\omega_0,0)&=\frac{\partial M_{rr}}{\partial\omega}(0,0,\omega_0,0)=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \\ \frac{\partial M_r}{\partial\mu}(0,\omega_0,0)&=\frac{\partial M_{rr}}{\partial\mu}(0,0,\omega_0,0)=\begin{bmatrix} \operatorname{Re} c_\mu & -\operatorname{Im} c_\mu \\ \operatorname{Im} c_\mu & \operatorname{Re} c_\mu \end{bmatrix}. \end{aligned}$$

Equation (104) is a system of two equations with four unknowns ($r=(\alpha,\beta)$, ω and μ). We now fix one of the unknowns setting

$$\alpha=0$$

such that we can expect one-parametric families of solutions (β,ω,μ). Introducing M_β as the second column of M_r and dropping the dependence on α (which is zero), the first derivative of $M_\beta(\beta,\omega,\mu)$ in $(0,\omega_0,0)$ with respect to the pair ω and μ is:

$$\begin{bmatrix} \dfrac{\partial M_\beta}{\partial\omega} & \dfrac{\partial M_\beta}{\partial\mu} \end{bmatrix}(0,\omega_0,0)=\begin{bmatrix} -1 & -\operatorname{Im} c_\mu \\ 0 & \operatorname{Re} c_\mu \end{bmatrix},$$

which is regular (as established in (98), since $\operatorname{Re} c_\mu\neq 0$ due to the assumption that the eigenvalue crosses the imaginary axis transversally). Note that M_β itself is a projection of

the first derivative of the original right-hand side of the full algebraic system (89). Thus, M_β is $k-1$ times continuously differentiable, and we end up with a system of two equations for three scalar variables (β, ω, μ):

$$0 = M_\beta(\beta, \omega, \mu)\,\beta.$$

Hence, either $\beta = 0$, which corresponds to the trivial solution or (after division by β)

$$0 = M_\beta(\beta, \omega, \mu), \tag{105}$$

where $M_\beta(0, \omega_0, 0) = (0, 0)$ and the derivative with respect to the pair (ω, μ) is regular in $(0, \omega_0, 0)$. Thus, we can apply the Implicit Function Theorem to (105) to obtain a unique graph $(\omega(\beta), \mu(\beta))$ solving (105). The graph satisfies $(\omega(0), \mu(0)) = (\omega_0, 0)$, and, thus, branches off from the trivial solution (which has $\beta = 0$ and ω and μ arbitrary). The rotational symmetry of M_r implies reflection symmetry of M_β in β such that $M_\beta(-\beta, \omega, \mu) = M_\beta(\beta, \omega, \mu)$ for all β, ω and μ. Hence, the solution graph is reflection symmetric, too: $\omega(-\beta) = \omega(\beta)$ and $\mu(-\beta) = \mu(\beta)$. Thus, for small β there is a unique non-trivial solution of the full algebraic system of the form $r = (0, \beta)$, $q = q(r, \omega(\beta), \mu(\beta))\,r$. As Equation (99) shows, we can extract the coordinates α (which is zero) and β from the full solution $x \in C^k(\mathbb{T}; \mathbb{R}^n)$ by applying the projections

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t) v_r^T x(t) - \sin(t) v_i^T x(t)\,\mathrm{d}t &= \frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{Re}\left[v_1 \exp(it)\right]^T x(t)\,\mathrm{d}t = \alpha,\\
\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(t) v_r^T x(t) + \cos(t) v_i^T x(t)\,\mathrm{d}t &= \frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{Im}\left[v_1 \exp(it)\right]^T x(t)\,\mathrm{d}t = -\beta,
\end{aligned}$$

which determines the First Fourier coefficients of x as claimed in (27) in Theorem 3.2. (Recall that the vector $v_1 = v_r + v_i$ was scaled to have unit length and that the decomposition was orthogonal.) □

References

[1] E. Allgower and M. K. Georgi. *Introduction to Numerical Continuation Methods*. SIAM, 1979.

[2] J. Baumgarte. Stabilization of constraints and integrals of motion in dynamical systems. *Computer Methods in Applied Mechanics and Engineering*, 1(1):1–16, 1972.

[3] E.A. Coddington and N. Levinson. *Theory of ordinary differential equations*. International series in pure and applied mathematics. McGraw-Hill, New York, 1955.

[4] O. Diekmann, S.A. van Gils, S.M. Verduyn Lunel, and H.-O. Walther. *Delay equations*, volume 110 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1995.

[5] Markus Eichmann. *A local Hopf Bifurcation Theorem for differential equations with state-dependent delays*. PhD thesis, University of Giessen, 2006.

[6] K. Engelborghs, T. Luzyanina, and D. Roose. Numerical bifurcation analysis of delay differential equations using DDE-BIFTOOL. *ACM Transactions on Mathematical Software*, 28(1):1–21, 2002.

[7] L. H. Erbe, W. Krawcewicz, and K. Geba. S^1-degree and global bifurcation theory of functional-differential equations. *J. Differ. Eq.*, 98:277–298, 1992.

[8] M. Golubitsky, D.G. Schaeffer, and I. Stewart. *Singularities and groups in bifurcation theory*, volume 69 of *Applied Mathematical Sciences*. Springer, New York, 1988.

[9] J. Guckenheimer and P. Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, volume 42 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1990.

[10] S. Guo. Equivariant Hopf bifurcation for functional differential equations of mixed type. *Applied Mathmeatics Letters*, 24:724–730, 2011.

[11] J.K. Hale and S.M. Verduyn Lunel. *Introduction to functional-differential equations*, volume 99 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1993.

[12] F. Hartung, T. Krisztin, H.-O. Walther, and J. Wu. Functional differential equations with state-dependent delays: theory and applications. In P. Drábek, A. Cañada, and A. Fonda, editors, *Handbook of Differential Equations*: *Ordinary Differential Equations*, volume 3, chapter 5, pages 435–545. North-Holland, 2006.

[13] Q. Hu and J. Wu. Global Hopf bifurcation for differential equations with state-dependent delay. *Journal of Differential Equations*, 248(10):2801–2840, 2010.

[14] A. R. Humphries, O. DeMasi, F. M. Magpantay, and F. Upham. Dynamics of a delay differential equation with multiple state dependent delays. *Discrete and Continuous Dynamical Systems A*, to appear.

[15] T. Insperger, D. A. W. Barton, and G. Stépán. Criticality of Hopf bifurcation in state-dependent delay model of turning processes. *International Journal of Non-Linear Mechanics*, 43(2):140 – 149, 2008.

[16] T. Insperger, G. Stépán, and J. Turi. State-dependent delay model for regenerative cutting processes. In *Proceedings of the Fifth EUROMECH Nonlinear Dynamics Conference, Eindhoven, Netherlands*, 2005.

[17] D. Jackson. *The Theory of Approximation*, volume XI. AMS Colloquium Publication, New York, 1930.

[18] W. Krawcewicz and J. Wu. *Theory of degrees with applications to bifurcations and differential equations*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons Inc., New York, 1997.

[19] T. Krisztin. A local unstable manifold for differential equations with state-dependent delay. *Discrete Contin. Dynam. Systems*, 9(4):993–1028, 2003.

[20] Y. A Kuznetsov. *Elements of Applied Bifurcation Theory*, volume 112 of *Applied Mathematical Sciences*. Springer-Verlag, New York, third edition, 2004.

[21] J. Mallet-Paret, R. D. Nussbaum, and P. Paraskevopoulos. Periodic solutions for functional differential equations with multiple state-dependent delays. *Topological Methods in Nonlinear Analysis*, 3(2):101–162, 1994.

[22] G Stépán. *Retarded Dynamical Systems: Stability and Characteristic Functions*. Longman Scientific and Technical, Harlow, Essex, 1989.

[23] H.-O. Walther. Stable periodic motion of a system with state-dependent delay. *Differential and Integral Equations*, 15:923–944, 2002.

[24] H.-O. Walther. Smoothness properties of semiflows for differential equations with state-dependent delays. *Journal of Mathematical Sciences*, 124(5):5193–5207, 2004.

[25] E. Winston. Uniqueness of the zero solution for differential equations with state-dependent delay. *J. Diff. Eqs.*, 7:395–405, 1970.

[26] J. Wu. Symmetric functional differential equations and neural networks with memory. *Transactions of the AMS*, 350(12):4799–4838, 1998.

A. Basic differentiability properties of the right-hand side

Let J be a compact interval or $\mathbb{T}$. Let $(D, \|\cdot\|_D)$ be a Banach space of the form

$$D = C^{k_1}(J;\mathbb{R}^{m_1}) \times \ldots \times C^{k_\ell}(J;\mathbb{R}^{m_\ell})$$

where $\ell \geq 1$, the integers k_j are non-negative and the integers m_j are positive. We use the natural maximum norm on the product D:

$$\|x\|_D = \|(x_1,\ldots,x_\ell)\|_D = \max_{j\in\{1,\ldots,\ell\}} \|x_j\|_{k_j},$$

and use the notation

$$\begin{aligned} D^k &= C^{k_1+k}(J;\mathbb{R}^{m_1}) \times \ldots \times C^{k_\ell+k}(J;\mathbb{R}^{m_\ell}), & \|x\|_{D,k} &= \max_{0\leq j\leq k} \|x^{(j)}\|_D, \\ D^{0,1} &= \{x \in D : L(x) < \infty\}\text{, with the norm} & \|x\|_{D,L} &= \max\left\{\|x\|_D, L(x)\right\}, \end{aligned}$$

where $x^{(j)}$ is the component-wise jth derivative and the Lipschitz constant $L(x)$ is defined as

$$L(x) = \sup_{t\neq s} \max_{j=1\ldots,\ell} \frac{|x_j^{(k_j)}(t) - x_j^{(k_j)}(s)|}{|t-s|},$$

where t and s in the index of sup are taken from J, if J is a compact interval, and from $\mathbb{R}$ if $J = \mathbb{T}$. Balls that are closed and bounded in $D^{0,1}$ are complete with respect to the norm of D.

A.1. Basic properties of f

This section proves three properties that EC^1 smooth functionals f have: first that the derivative limit (10) exists also for Lipschitz continuous deviations, second a weaker form of the mean value theorem, and third, local EC Lipschitz continuity.

Lemma A.1 (Extension of derivative to deviations in $D^{0,1}$)

> Let $f : D \mapsto \mathbb{R}^n$ be EC^1 smooth in the sense of Definition 2.1. Then the limit required to exist in Definition2.1 exists also in the $\|\cdot\|_{D,L}$-norm: for all $x \in D^1$
>
> $$\lim_{\substack{y \in D^{0,1} \\ \|y\|_{D,L} \to 0}} \frac{|f(x+y) - f(x) - \partial^1 f(x,y)|}{\|y\|_{D,L}} = 0. \tag{106}$$

Note that in (106) the norm in which y goes to zero is $\|\cdot\|_{D,L}$ instead of $\|\cdot\|_{D,1}$.

Proof This is a simple continuity argument. Let $\varepsilon > 0$ be arbitrary. We pick $\delta > 0$ such that

$$|f(x+\tilde{y}) - f(x) - \partial^1 f(x,\tilde{y})| < \varepsilon \|\tilde{y}\|_{D,1} \tag{107}$$

for all $\tilde{y} \in D^1$ satisfying $\|\tilde{y}\|_{D,1} < \delta$. Let $y \in D^{0,1}$ be such that $\|y\|_{D,L} < \delta$. We can choose a $\tilde{y} \in D^1$ that satisfies

$$\|\tilde{y}\|_{D,1} < \min\{\delta, 2\|y\|_{D,L}\} \tag{108}$$

$$|f(x+y) - f(x+\tilde{y})| < \varepsilon \|y\|_{D,L} \tag{109}$$

$$|\partial^1 f(x, y - \tilde{y})| < \varepsilon \|y\|_{D,L}. \tag{110}$$

Condition (108) can be achieved because D^1 is a dense subspace in $D^{0,1}$, and for every element $\tilde{y}$ of D^1 the $\|\cdot\|_{D,1}$-norm is not larger than the $\|\cdot\|_{D,L}$-norm: $\|\tilde{y}\|_{D,1} \leq \|\tilde{y}\|_{D,L}$. (109) follows from the continuity of f and the density of $D^{0,1}$ in D^1, and (110) follows from the continuity of $\partial^1 f$ as a map on $D^1 \times D$, and the density of $D^{0,1}$ in D^1. Combining estimate (107) with (108)–(110) we obtain

$$|f(x+y) - f(x) - \partial^1 f(x,y)| < 4\varepsilon \|y\|_{D,L}.$$

□

Lemma A.2 (Existence of mean value)

> There exists a continuous function
>
> $$\tilde{a} : D^1 \times D^1 \times D \mapsto \mathbb{R}^n$$

which is linear in its third argument and satisfies for all $x, y \in D^1$

$$f(x+y) - f(x) = \tilde{a}(x, y, y). \tag{111}$$

Moreover, $\tilde{a}(x, 0, y) = \partial^1 f(x, y)$ for all $x \in D^1$ and $y \in D$.

Proof The argument for the existence of a mean value follows exactly the proof of the general mean value theorem [12]: the candidate for $\tilde{a}(u, v, w)$ is

$$\tilde{a}(u, v, w) = \int_0^1 \partial^1 f(u + sv, w)\, \mathrm{d}s. \tag{112}$$

Since $\partial^1 f$ is assumed to be continuous in its arguments the integral is well defined and continuous in its arguments $u \in D^1$, $v \in D^1$, $w \in D$. All one has to show is that the $\tilde{a}$ defined in (112) satisfies (111): let $x, y \in D^1$ and $\varepsilon > 0$ be arbitrary, and choose m such that uniformly for all $s \in [0, 1]$

$$\left| \int_0^1 \partial^1 f(x + sy, y)\, \mathrm{d}s - \frac{1}{m} \sum_{k=0}^{m-1} \partial^1 f\left(x + \frac{k}{m} y, y\right) \right| < \varepsilon,$$

$$\left| f\left(x + sy + \frac{y}{m}\right) - f(x + sy) - \partial^1 f\left(x + sy, \frac{y}{m}\right) \right| < \frac{\varepsilon}{m}.$$

Then it follows that

$$\left| f(x+y) - f(x) - \int_0^1 \partial^1 f(x + sy, y)\, \mathrm{d}s \right| < 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary the left-hand side must be zero. □

Lemma A.3 (Local EC Lipschitz continuity)

For all $x \in D^{0,1}$ there exists a neighborhood $U(x) \subseteq D^{0,1}$ and a constant $K_x > 0$ such that for all y_1 and $y_2 \in U(x)$ the following Lipschitz estimate holds:

$$|f(y_1) - f(y_2)| \le K_x \|y_1 - y_2\|_D.$$

Note that the upper bound depends only on the $\|\cdot\|_D$-norm, not on the $\|\cdot\|_{D,L}$-norm, which would be a weaker statement.

Proof We prove the Lipschitz continuity first for y_1 and y_2 from a sufficiently small neighborhood $U(x) \cap D^1 \subseteq D^1$ of $x \in D^1$.

Let x be an element of D^1. Since the mean value $\tilde{a}$ is continuous in $(x, 0, 0)$, and $\tilde{a}(x, 0, 0) = 0$, we have a $\delta > 0$ such that for all $u, v \in D^1$ and $w \in D$ satisfying $\|u\|_{D,1} < \delta$, $\|v\|_{D,1} < \delta$ and $\|w\|_D < \delta$

$$|\tilde{a}(x + u, v, w)| < \varepsilon.$$

This implies that $|\tilde{a}(x+u,v,w)| < [\varepsilon/\delta]\|w\|_D$ for u and v with $\max\{\|u\|_{D,1},\|v\|_{D,1}\} < \delta$ and $w \in D$ (since $\tilde{a}$ is linear in its third argument). Thus, $\|\tilde{a}(x+u,v,\cdot)\|_D \leq \varepsilon/\delta$ for $\tilde{a}(x+u,v,\cdot)$ as an element of $L(D;D)$ in the operator norm corresponding to D. Consequently, if $\|y_1 - x\|_{D,1} < \delta/2$ and $\|y_2 - x\|_{D,1} < \delta/2$

$$|f(y_1) - f(y_2)| = \left| \int_0^1 \tilde{a}(y_2, y_1 - y_2, y_1 - y_2)\,\mathrm{d}s \right| \leq \frac{\varepsilon}{\delta}\|y_1 - y_2\|_D,$$

such that we can choose $K_x = \varepsilon/\delta$. The extension of the statement to $D^{0,1}$ follows from the continuity of f in D: $U(x_0) \cap D^1$ is dense in $U(x_0) \subset D^{0,1}$ using the $\|\cdot\|_{D,L}$-norm. Pick two sequences y_n and z_n in $U(x_0) \cap D^1$ that converge to y and z in $U(x_0)$ in the Lipschitz norm. Then $f(y_n) \to f(y)$ and $f(z_n) \to f(z)$ since f is continuous in D. Moreover, $\|y_n - z_n\|_D \to \|y - z\|_D$ for $n \to \infty$. Since

$$|f(y_n) - f(z_n)| \leq K_x \|y_n - z_n\|_D \tag{113}$$

for all n the inequality also holds for the limit for $n \to \infty$. □

A.2. Basic properties of F

In this section we restrict ourselves to the periodic case: $J = \mathbb{T}$. Let $F : D \mapsto C^0(\mathbb{T};\mathbb{R}^n)$ be defined as $F(x)(t) = f(\Delta_t x)$.

Lemma A.4 (Continuity of F)

> Let $f : D \mapsto \mathbb{R}^n$ be continuous. Then $F : D \mapsto C^0(\mathbb{T};\mathbb{R}^n)$ is also continuous.

Proof This is a simple consequence of the continuity of f, the continuity of $(t,x) \mapsto \Delta_t x$ with respect to both arguments (t and x) in the $\|\cdot\|_0$-norm, and the compactness of $\mathbb{T}$. Let $\varepsilon > 0$ and $x \in D$ be arbitrary. We want to prove continuity of F in x. So, we have to find a $\delta > 0$ such that

$$\left| f(\Delta_s x + h) - f(\Delta_s x) \right| < \varepsilon \quad \text{for all } s \in \mathbb{T} \text{ and } h \in D, \text{ satisfying } \|h\|_D < \delta. \tag{114}$$

(Since $\|\Delta_s h\|_D = \|h\|_D$ we can replace $\Delta_s h$ by h.) The continuity of f implies that for every $r > 0$ and every $t \in \mathbb{T}$ we find a $\delta_x(t,r)$ such that

$$|f(\Delta_t x + h) - f(\Delta_t x)| < r \quad \text{whenever } \|h\|_D < \delta_x(t,r). \tag{115}$$

For every $t \in \mathbb{T}$ there exists an open neighborhood $U(t) \subset \mathbb{T}$ such that

$$\|\Delta_s x - \Delta_t x\|_D < \delta_x(t,\varepsilon/2)/2 \quad \text{for all } s \in U(t),$$

because the function $t \in \mathbb{T} \mapsto \Delta_t x$ is continuous in t. These neighborhoods $U(t)$ are an open cover of the compact set $\mathbb{T}$, so there exist finitely many $t_1, \ldots, t_m \in \mathbb{T}$ such that the union of the neighborhoods $U(t_j)$ contains all points $s \in \mathbb{T}$. We choose

$$\delta = \min_{j=1,\ldots,m} \delta_x(t_j,\varepsilon/2)/2,$$

which is a positive quantity. Let $s \in \mathbb{T}$ be arbitrary and let $h \in D$ satisfy $\|h\|_D < \delta$. We have to check the inequality (114). The point s is in one of the neighborhoods $U(t_j)$, say without loss of generality, $s \in U(t_1)$. Thus, $\|\Delta_s x - \Delta_{t_1} x\|_D < \delta_x(t_1, \varepsilon/2)/2$, and, consequently, $\|\Delta_s x - \Delta_{t_1} x + h\|_D < \delta_x(t_1, \varepsilon/2)$ (because also $\|h\|_D < \delta \le \delta_x(t_1, /\varepsilon/2)/2$). Therefore, we can split up the difference $|f(\Delta_s x + h) - f(\Delta_s x)|$:

$$\begin{aligned} |f(\Delta_s x + h) - f(\Delta_s x)| \le & \left| \left[f\left(\Delta_{t_1} x + (\Delta_s x - \Delta_{t_1} x + h)\right) - f(\Delta_{t_1} x) \right] \right| \\ & + \left| \left[f\left(\Delta_{t_1} x + (\Delta_s x - \Delta_{t_1} x)\right) - f(\Delta_{t_1} x) \right] \right| \\ & < \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

Note that the deviations from $\Delta_{t_1} x$ in the arguments of f in both terms of the sum are less than or equal to $\delta_x(t_1, \varepsilon/2)$ such that we can apply (115) for $t = t_1$, $r = \varepsilon/2$. □

The following lemma lists properties that F has if f satisfies local EC Lipschitz continuity in the sense of Definition 2.2. That is, we do *not* assume that f is EC^1 smooth in the sense of Definition 2.1 for Lemma A.5. Since Lemma A.3 was proved using only the assumption of EC^1 smoothness of f, local EC Lipschitz continuity is a weaker condition.

Lemma A.5 (EC Lipschitz continuity of F)

Assume that $f : D \mapsto \mathbb{R}^n$ is locally EC Lipschitz continuous in the sense of Definition 2.2. Then F has the following properties:

1. for all $x \in D^{0,1}$ there exists a neighborhood $U(x) \subseteq D^{0,1}$ and a constant $K_x > 0$ such that for all y_1 and $y_2 \in U(x)$
$$\|F(y_1) - F(y_2)\|_0 \le K_x \|y_1 - y_2\|_D.$$
2. F maps elements of $D^{0,1}$ into $C^{0,1}(\mathbb{T}; \mathbb{R}^n)$. Moreover, for every $x \in D^{0,1}$, any bounded neighborhood $U(x) \subseteq D^{0,1}$ for which the Lipschitz constant K_x exists has a bounded image under F: there exists a bound $R > 0$ such that $\|F(y)\|_{0,1} \le R$ for all $y \in U(x)$ (R depends on $U(x)$).

Proof Statement 1 is a consequence of the local EC Lipschitz continuity of f and the compactness of $\mathbb{T}$ (which allows one to choose a uniform Lipschitz bound K_x for all $t \in \mathbb{T}$).

Concerning statement 2: let $x \in D^{0,1}$ be arbitrary, and let the neighborhood $U(x)$ be bounded (say, $\|y - x\|_{D,L} \le \delta$) such that F has a Lipschitz constant K_x in $U(x)$. Then we have for all $y, z \in U(x)$ and $t, s \in \mathbb{T}$ the estimate

$$|f(\Delta_t y) - f(\Delta_s z)| \le K_x \|\Delta_t y - \Delta_s z\|_D = K_x \|\Delta_{t-s} y - z\|_D.$$

Initially setting $z = x$ and $s = t$ we get a bound on $\|F(y)\|_0$: $\|F(y)\|_0 \le \|F(x)\|_0 + K_x \delta =: R_0$ for all $y \in U(x)$. It remains to be shown that the Lipschitz constant of $F(y)$ is bounded for $y \in U(x)$:

$$|F(y)(t) - F(y)(s)| = |f(\Delta_t y) - f(\Delta_s y)| \le K_x \|\Delta_t y - \Delta_s y\|_D \le K_x \|y\|_{D,L} |t - s|.$$

Since $\|y - x\|_{D,L} \leq \delta$ for $y \in U(x)$, choosing

$$R = \max\left\{R_0, K_x\left(\|x\|_{D,L} + \delta\right)\right\}$$

ensures that $\|F(y)\|_{0,1} \leq R$. □

Define the maps

$$\begin{aligned} \partial^1 F(u,v)(t) &= \partial^1 f(\Delta_t u, \Delta_t v) && \text{for } u \in D^1,\ v \in D, \\ \tilde{A}(u,v,w)(t) &= \tilde{a}(\Delta_t u, \Delta_t v, \Delta_t w) && \text{for } u \in D^1,\ v \in D^1,\ w \in D. \end{aligned}$$

The following Lemma A.6, and Lemma A.7 assume that f is EC^1 smooth in D in the sense of Definition 2.1.

Lemma A.6 (Differentiability of F)

Let $f : D \mapsto \mathbb{R}^n$ be EC^1 smooth. Then F, $\partial^1 F$ and $\tilde{A}$ have the following properties.

1. The map $(u,v) \mapsto \partial^1 F(u,v)$ is continuous in both arguments (and linear in its second argument) as a map from $D^1 \times D$ into $C^0(\mathbb{T};\mathbb{R}^n)$. It satisfies for all $x \in D^1$
$$\lim_{\substack{y \in D^{0,1} \\ \|y\|_{D,L} \to 0}} \frac{\|F(x+y) - F(x) - \partial^1 F(x,y)\|_0}{\|y\|_{D,L}} = 0. \tag{116}$$

2. The map $\tilde{A}(u,v,w)$ is continuous in all three arguments (and linear in its third argument) as a map from $D^1 \times D^1 \times D$ into $C^0(\mathbb{T};\mathbb{R}^n)$. It satisfies for all $x, y \in D^1$
$$F(x+y) - F(x) = \tilde{A}_1(x,y,y).$$
Moreover, $\tilde{A}(x,0,y) = \partial^1 F(x,y)$ for all $x \in D^1$ and $y \in D$.

Note that in the limit (116) we allow for deviations $y \in D^{0,1}$.

Proof The continuity of $\partial^1 F$ follows from the continuity of $\partial^1 f$ by applying Lemma A.4 to $\partial^1 f : D^1 \times D \mapsto \mathbb{R}^n$ instead of f. The linearity of $\partial^1 F$ in its second argument follows from the linearity of $\partial^1 f$ in its second argument.

The limit (116) also follows from the corresponding limit (106): let $x \in D^1$ and $\varepsilon > 0$ be arbitrary. For every fixed t there exists a $\delta(t) > 0$ such that

$$\frac{|f(\Delta_t x + \Delta_t y) - f(\Delta_t x) - \partial^1 f(\Delta_t x, \Delta_t y)|}{\|y\|_{D,L}} < \varepsilon \tag{117}$$

for all y with $\|y\|_{D,L} < \delta(t)$. As f and $\partial^1 f$ are continuous in their arguments $x \in D^1$ and $y \in D^{0,1}$, the inequality also holds for all s in a sufficiently small open neighborhood of t, $U(t)$. The set of neighborhoods $U(t)$ for all $t \in \mathbb{T}$ are a cover of the compact set $\mathbb{T}$ by open sets. Choosing a finite subcover from this cover, and labeling the times $t_1, \ldots, t_m$, we can choose

$$\delta = \min_{k=1,\ldots,m} \delta(t_k)$$

such that (117) holds for all uniformly $t \in \mathbb{T}$. This proves statement 1 of the lemma.

Concerning statement 2: for the continuity of $\tilde{A}$ we invoke again Lemma A.4, this time for $\tilde{a}$ on $D^1 \times D^1 \times D$. The linearity of $\tilde{A}$ in its third argument follows from the linearity of $\tilde{a}$ in its third argument. The relations $F(x+y)(t) - F(x)(t) = \tilde{A}(x,y,y)(t)$ and $\tilde{A}(x,0,y)(t) = \partial^1 F(x,y)(t)$ in every $t \in \mathbb{T}$ follow from the corresponding relations for f and $\tilde{a}$, as stated in Lemma A.2. □

Lemma A.7 (Differentiability of images of F)

> Let $f : D \mapsto \mathbb{R}^n$ be EC^1 smooth and let $k \geq 0$ be some integer. We assume that $F : D \mapsto C^k(\mathbb{T};\mathbb{R}^n)$ and $\partial^1 F : D^1 \times D \mapsto C^k(\mathbb{T};\mathbb{R}^n)$ are continuous maps. Then F maps elements of D^1 into $C^{k+1}(\mathbb{T};\mathbb{R}^n)$, and F is continuous as a map from D^1 to $C^{k+1}(\mathbb{T};\mathbb{R}^n)$.

Proof Let x be in D^1, that is, $x' \in D$. If $\partial^1 F : D^1 \times D \mapsto C^k(\mathbb{T};\mathbb{R}^n)$ is continuous then $\tilde{A} : D^1 \times D^1 \times D \mapsto C^k(\mathbb{T};\mathbb{R}^n)$, which is given by $\tilde{A}(u,v,w) = \int_0^1 \partial^1 F(u+sv,w)\,\mathrm{d}s$, is continuous, too. Using statement 2 of Lemma A.6 we have

$$\frac{F(\Delta_h x) - F(x)}{h} = \tilde{A}\left(x, \Delta_h x - x, \frac{\Delta_h x - x}{h}\right). \tag{118}$$

On the right side $\|\Delta_h x - x\|_{D,1}$ converges to 0 for $h \to 0$. Also,

$$\left\|\frac{\Delta_h x - x}{h} - x'\right\|_D \to 0 \quad \text{for } h \to 0,$$

because $x \in D^1$. Since $\tilde{A}$ is continuous in its arguments the right side converges to $\tilde{A}(x,0,x') = \partial^1 F(x,x') \in C^k(\mathbb{T};\mathbb{R}^n)$ for $h \to 0$. Thus, the limit of the left-hand side in (118) for $h \to 0$ exists, too, such that $F(x) \in C^{k+1}(\mathbb{T};\mathbb{R}^n)$ and the time derivative $(F(x))'$ equals $\partial^1 F(x,x')$. Since $(v,w) \in D^1 \times D \mapsto \partial^1 F(v,w) \in C^k(\mathbb{T};\mathbb{R}^n)$ is continuous in (u,v), the time derivative of $F(x)$, $(F(x))' = \partial^1 F(x,x')$ is also continuous in x if we use the norm $\|\cdot\|_{D,1}$ for the argument and $\|\cdot\|_k$ for the image. □